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Consumer's Optimal Choice on l_p Commodity Space*

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Abstract Some results about consumer optimal choice based on commodity space $l_p(1 \leq p < +\infty)$ are given by using the weak Kolmogorov characteristic (WKC), which is introduced in this paper. At the same time, economical meanings of WKC and upper-semi continuity of optimal-equilibrium-choice operator are also discussed.

Keywords Best equilibrium; Weak Kolmogorov Characteristic; upper-semi continuity; Commodity space

1 Introduction

Traditional economics always investigates economic phenomenons in the commodity space which has finite kinds of commodities. However economic phenomenons are extremely complicated, economic behaviors, in fact, arise in infinity dimensional space. So functional analysis, the mathematic tool plays more important roles in economic analysis than ever before. Some researches have been done in this field, such as [4, 6, 5, 7]. T. F. Bewley has applied $l_p(1 \leq p < +\infty)$ to characterize economic behavior, and space L_p has also been used to analyze financial behavior on the condition of uncertainty[8]. Methods used in traditional economics can not meet the needs of the research in infinity commodity space, which lead people to find new ways. In this paper, We apply Kolmogorov characteristic in best approximation theory to depict the consumer's optimal choice, that is to say, the consumer's optimal equilibrium in the commodity space $l_p(1 \leq p < +\infty)$.

The rest of this paper proceeds as follows. In section 2, we give some basic assumption about consumer. Section 3 includes some notations and their properties. The content of the section 4 presents consumer's optimal choice, which is pictured by weak Kolmogorov characteristic. The future work is given in the section 5.

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2 Basic Assumption about Consumer

We assume there are infinity kinds of commodities, and let $x = (x_1, x_2, \cdots, x_n, \cdots)$ be a commodity combination, in which x_i is the ith commodity and its size can't be less than zero. Thus we can investigate the problems on l_p^+, the positive part of the space $l_p(1 \le p < +\infty)$. Let $\|x\|_p = (\sum_{i=1}^{\infty} |x_i|^p)^{\frac{1}{p}}$ be the norm of x, which is always used to show consumer's behaviors in economics, then we assume

$$l_p^+ = \{x = (x_1, x_2, \cdots, x_n, \cdots) | x_i \ge 0, \|x\|_p < +\infty\}.$$

Let p_i be the price of the ith commodity in a certain market, and $P = (p_1, p_2, \cdots, p_n, \cdots)$ be the price vector. If $x \in l_p^+$, then the inner product $\langle P, x\rangle = \sum_{i=1}^{\infty} p_i x_i$ is the value of the commodity combination x. In fact, $P \in l_q^+$, where l_q is the dual space of $l_p(\frac{1}{p} + \frac{1}{q} = 1)$, and its norm can be defined as $\|P\|_q = (\sum_{i=1}^{\infty} |p_i|^q)^{\frac{1}{q}}$. When investigating a consumer economic behavior, price vector is used to be standardized, $\|P\|_q = 1$, for it makes less impact on establishing conclusion, so we let

$$l_q^+ = \{P = (p_1, p_2, \cdots, p_n, \cdots) | p_i \ge 0, \|P\|_q = 1\}.$$

The following assumptions are given before analyzing the consumer's economic behavior in the commodity space $l_p(1 \le p < +\infty)$.

(1) Consumption of each commodity used by every consumer is a little, which can not make price to fluctuate, that is to say, all the consumers are the accepters of the price system.

(2) Every consumers is rational. This means that his or her budgets for purchasing commodities in l_p^+ can meet his demand without wasting money. They like low price and abhor high price for the same commodity combination.

(3)A consumer preference ($\preceq$ or $\succeq$) is a binary relation with reflexivity, transitivity and completeness, and it is continuous, discontented and strictly convex [9].

(4) A consumer not only has some monetary wealth M, but also some material wealth X in the price system P_m. Let $E = (P_m, M, X)$ be the present market circumstance and consumer's economic- condition, which is called economic-condition simply.

3 Some Notations and Their Properties

A consumer consumes a commodity-combination Z_0 on the economic condition $E = (P_m, M, X)$. Let $I_E(Z_0) = \{y | y \sim Z_0, y \in l_p^+\}$ be the set of all the combination which are able to meet his same demand as Z_0 does, we call it as indifference-surface, where $y \sim z$ presents a combination y can meets the consumer's same demand as combination z does. Let

$$W_E = \{y : y \succeq Z, y \in l_p^+, Z \in I_E(Z_0)\}$$

be the desire set , and

$$B_E = \{y : \langle P_m, y\rangle \le M, y \in l_p^+\}$$

the purchasable-commodity-combination set of a consumer on the economic condition of $E = (P_m, M, X)$, simply we call it as purchasable set.

Since consumer is rational, on the economic condition of $E = (P_m, M, X)$, he thinks that the material wealth X he possess can not meet his desire of $Z_0 \in l_p^+$, namely $X \preceq Z_0$. If he expects to consume a commodity-combination which can meet same demand as Z_0 does, he have to pay extra money. Let

$$T = \{y : \langle P_m, y\rangle < \inf_{g \in W_E} \langle P_m, g\rangle, y \in l_p^+\},$$

then $X \in T$.

Proposition 1. *W_E is closed and convex set.*

Proof. For $\forall y_1, y_2 \in W_E$, then we have $y_1 \succeq Z_0, y_2 \succeq Z_0$, because of the completeness of consumer preference. We get $\alpha y_1 + (1-\alpha)y_2 \succ y_2 \succeq Z_0$, for preference '$\succeq$' has the strictly convexity. In addition, Closeness of W_E is due to the continuity of preference '$\succeq$'. □

Proposition 2. *B_E is a convex and closed set.*

Proof. For $\forall y_1, y_2 \in B_E$, then we have $\langle P_m, y_1\rangle \le M, \langle P_m, y_2\rangle \le M$ by the definition of B_E. For any $\alpha \in [0,1]$, $\langle P_m, (\alpha y_1 + (1-\alpha)y_2\rangle = \alpha\langle P_m, y_1\rangle + (1-\alpha)\langle P_m, y_2\rangle \le M$, which infers that B_E is a convex set.

Let $y' \in B'_E$, where B'_E is the derived set of B_E (namely, the accumulation point set of B_E). If y' is not in B_E, then $\langle P_m, y'\rangle > M$. Let

$$\varepsilon = \inf_{y \in B_E} \langle P_m, y' - y\rangle > 0,$$

and

$$U(y', \varepsilon) = \{u : \|u - y'\|_p < \varepsilon, u \in l_p^+\}$$

be the epsilon neighborhood of y'. Since y' is the accumulation point of B_E, there exists a $u_0 \in U(y', \varepsilon)$ such that $u_0 \ne y'$. Then we have

$$\|u_0 - y'\|_p < \varepsilon = \inf_{y \in B_E} \langle P_m, y' - y\rangle,$$

but

$$|\langle P_m, u_0 - y'\rangle| \le \|u_0 - y'\|_p < \varepsilon = \inf_{y \in B_E} \langle P_m, y' - y\rangle \le \langle P_m, y' - u_0\rangle,$$

which is a contradition. So we have $y' \in B_E$, moreover, B_E is a closed set. □

Combining proposition 1 with proposition 2, we know that W_E and B_E are closed set with strong topology[10].

Definition 1. Let $C_E = B_E \cap W_E$, we call C_E as consumable set of a consumer on the condition of $E = (P_m, M, X)$.

From the proposition1 and 2, the following proposition is obtained.

Proposition 3. *C_E is a convex and closed set.*

If C_E is an empty set, it is evident that the consumer has more desire than his paying ability. Hence we only investigate the case that C_E is nonempty, that is to say, C_E not only meets consumer's needs, but also does not exceed his payment capacity.

4 Consumer's Optimal Equilibrium —-Weak Kolmogorov Characteristic

Assuming a consumer has material wealth $X \in T$, he expects to get the same satisfaction as consuming commodity combination Z_0, so he has to pay more money. Because of consumer's rationality, He will select his commodity-combination in C_E to get his satisfaction with money as little as possible. Assuming $y \in C_E$, let

$$F_E(y,X) = \langle P_m, y-X\rangle,$$

then $F_E(y,X)$ indicates the quantity of money which has been paid by a consumer for the commodity combination of $y-X$ on the economical condition of $E = (P_m,M,X)$. Specially, $X = 0$ means that the consumer does not have any material wealth currently, and he has to pay $F_E(y,0)$ for y. When $y = X$, $F_E(X,X) = \langle P_m, X-X\rangle = \langle P_m, 0\rangle = 0$.It is evident that the current material wealth has met his need, and he must not pay any money for extra commodities. Assuming $y \in C_E, X \in T$, we have $F_E(y,X) \geq 0$. If $X = 0$, then $F_E(y,0) > 0$ clearly.

Definition 2. Assuming a consumer economical condition to be $E = (P_m,M,X)$, if there exists a $y_0 \in C_E$, such that

$$F_E(y_0,X) = \inf_{y\in C_E} \langle P_m, y-X\rangle,$$

then we call y_0 to be the optimal equilibrium point of the consumer on the economical condition E, and all of them denote $P_{C_E}(X)$, which is called as the optimal equilibrium set of the consumer on the condition of E, and $F_E(y_0,X)$ is called optimal payment of the consumer on the condition E.

Definition 3. Assuming a consumer economical condition to be $E = (P_m,M,X)$, if there exists a $y_0 \in P_{C_E}(X)$, then we call $F_E(y,y_0) \geq 0 (y \in C_E)$ to be the Weak Kolmogorov characteristic (WKC) of the consumer on the economical condition E.[2, 3]

Remark 1. 'Kolmogorov characteristic' originates from functional analysis and best approximation theory. Because we use the condition in this paper is week to the best approximation theory, so we call it as the 'Weak Kolmogorov characteristic' (WKC).[1]

Remark 2. If a consumer has an economical condition $E = (P_m,M,X)$, the WKC has profound economic implication: on the condition of $E = (P_m,M,X)$, for the purpose of getting more satisfaction than which the present material wealth X provides, he has to pay more money, and the money he pays for the commodity combination $y-X$ is larger than that he pays for $y_0 - X$, which is $F_E(y,y_0)$, because

$$0 \leq F_E(y,y_0) = \langle P_m, y-y_0\rangle = \langle P_m, y-X\rangle - \langle P_m, y_0-X\rangle,$$

which indicates consuming $(y_0 - X)$ in place of $(y-X)$ not only has saved his money, but also satisfied himself, so the optimal equilibrium reaches . It is proved that the element $y_0 \in P_{C_E}(X)$ is able to meet the consumer's demand and spent minimum money at the same time. Thus, we call $P_{C_E}(X)$ to be the optimal equilibrium set of the consumer on the condition of E.

Remark 3. In fact, when X is a variable in T, market condition P_m and monetary wealth M are constants, $P_{C_E}(X)$ is a set-valued map of X, which is called optimal equilibrium choice operator.

Proposition 4. *Assuming a consumer's economical condition to be $E = (P_m, M, X)$, then $y_0 \in P_{C_E}(X)$, if and only if the consumer has the WKC on the economical condition E.*

Proof. "$\Longrightarrow$"

Assuming $X \in \bar{T}$, then $\langle P_m, X\rangle \leq \inf_{y \in W_E} \langle P_m, y\rangle$. From $y_0 \in P_{C_E}(X)$, we get

$$\langle P_m, y_0 - X\rangle = \inf_{y \in C_E} \langle P_m, y - X\rangle,$$

and

$$\langle P_m, y_0 - X\rangle \leq \langle P_m, y - X\rangle.$$

Therefore we have $\langle P_m, y - y_0\rangle \geq 0$, namely $F_E(y, y_0) \geq 0$.

"$\Longleftarrow$"

Assuming a consumer having the WKC on the economical condition of E, that is to say, $F_E(y, y_0) \geq 0$, and $y, y_0 \in C_E$.

Assuming y_0 not to be the best choice of the consumer on the condition of E, namely $y_0 \notin P_{C_E}(X)$, then

$$\langle P_m, y_0 - X\rangle > \inf_{y \in C_E} \langle P_m, y - X\rangle.$$

Let

$$\varepsilon = \langle P_m, y_0 - X\rangle - \inf_{y \in C_E} \langle P_m, y - X\rangle,$$

it is evident that $\varepsilon > 0$. From the definition of infimum, there exists a $y' \in C_E$, such that

$$\langle P_m, y' - X\rangle < \inf_{y \in C_E} \langle P_m, y - X\rangle + \varepsilon,$$

which infers $\langle P_m, y_0 - X\rangle \geq \langle P_m, y' - X\rangle$,namely $F_E(y_0, y') > 0$. It is contradictory to the WKC, hence $y_0 \in P_{C_E}(X)$. □

Corollary 5. *Assuming a consumer's economical condition to be $E_0 = (P_m, M, 0)$, then $y_0 \in P_{C_E}(0)$, if and only if the consumer has the WKC on the economical condition E_0.*

$X = 0$ means the consumer does not have any material wealth before buying commodity combination y_0.

Corollary 6. *Assuming a consumer's economical condition to be $E = (P_m, M, X)$, then the consumer's optimal equilibrium set $P_{C_E}(X)$ is a convex set.*

Proof. Assuming $y_0, y_1 \in P_{C_E}(X)$, then for any $y \in C_E$, we have $F_E(y, y_0) \geq 0$ and $F_E(y, y_1) \geq 0$ according to Proposition 4. For any $\alpha \in [0, 1]$, then

$$\begin{aligned} & F_E(y,(\alpha y_0+(1-\alpha)y_1) \\ = \;& \langle P_m, y-(\alpha y_0+(1-\alpha)y_1)\rangle \\ = \;& \alpha\langle P_m, y-y_0\rangle+(1-\alpha\langle P_m, y-y_1\rangle) \\ = \;& F_E(y,y_0)+F_E(y,y_1)\geq 0, \end{aligned}$$

so $\alpha y_0+(1-\alpha)y_1 \in P_{C_E}(X)$, which implies $P_{C_E}(X)$ is a convex set. □

Definition 4. Let P be a set-valued mapping from l_p to 2^{l_p}, namely, for any $X \in l_p$, there exists a set $P(X) \subset l_p$. For any neighborhood U of set $P(X_0) \subset l_p$, there exist a δ more than zero, if $||X-X_0||_p < \delta$, we have $P(X) \subset U$, then call the set-valued mapping to be upper semi-continuity at X_0.[10]

Proposition 7. *Assuming a consumer's economical condition to be $E = (P_m, M, X)$, if market price vector P_m and his monetary wealth M are constant, then his optimal equilibrium choice operator $P_{C_E}(X)$ is upper semi-continuity on T, where $X \in T$.*

Proof. Assuming $P_{C_E}(X)$ is not upper semi-continuity at $X_0 \in T$, then there exist a neighborhood U_0 of X_0 (U_0 is a open set, and $U_0 \subset C_E$) and $X_n \in T$, such that $P_{C_E}(X_0) \subset U_0, ||X_n-X_0||_p \to 0$, but $P_{C_E}(X_n) \not\subseteq U_0$, which implies that there is a $y_n \in P_{C_E}(X_n)$, such that $y_n \notin U_0$. let

$$A = \lim_{n\to\infty}\inf_{y\in C_E}\langle P_m, y-X_n\rangle,\;\; B = \inf_{y\in C_E}\langle P_m, y-X_0\rangle,$$

we assert $A = B$. if $A \neq B$, then $A > B$ is assumed without loss of generality. Let $\varepsilon = A-B$, then there is a $y' \in C_E$, such that

$$\langle P_m, y'-X_0\rangle < B+\varepsilon = A,$$

but we also have

$$A = \lim_{n\to\infty}\inf_{y\in C_E}\langle P_m, y-X_n\rangle \leq \lim_{n\to\infty}\langle P_m, y'-X_n\rangle = \langle P_m, y'-X_0\rangle$$

which implies

$$\langle P_m, y'-X_0\rangle < \langle P_m, y'-X_0\rangle.$$

It is a contradictory, which infers $A = B$.

Assuming $y_0 \in P_{C_E}(X_0)$, from

$$\begin{aligned} & \langle P_m, y_n-X_0\rangle \\ = \;& \langle P_m, y_n-X_n-X_n-X_0\rangle \\ \leq \;& \langle P_m, y_n-X_n\rangle+||X_n-X_0||_p \\ = \;& \inf_{y\in C_E}\langle P_m, y-X_n\rangle+||X_n-X_0||_p, \end{aligned}$$

$A = B$, and $\inf_{y \in C_E} \langle P_m, y - y_0 \rangle = 0$, we have

$$\begin{aligned}
& \langle P_m, y_0 - X_0 \rangle \\
= \ & \inf_{y \in C_E} \langle P_m, y - X_0 \rangle \\
\leq \ & \lim_{n \to \infty} \langle P_m, y_n - X_0 \rangle \\
\leq \ & \lim_{n \to \infty} (\langle P_m, y_n - X_n \rangle + \|X_n - X_0\|_P) \\
= \ & \lim_{n \to \infty} \langle P_m, y_n - X_n \rangle \\
= \ & \lim_{n \to \infty} \inf_{y \in C_E} \langle P_m, y - X_n \rangle \\
= \ & \lim_{n \to \infty} \langle P_m, y_0 - X_n \rangle + \lim_{n \to \infty} \inf_{y \in C_E} \langle P_m, y - y_0 \rangle \\
= \ & \langle P_m, y_0 - X_0 \rangle,
\end{aligned}$$

which implies $\lim_{n \to \infty} \langle P_m, y_n - X_0 \rangle = \langle P_m, y_0 - X_0 \rangle$, namely $\lim_{n \to \infty} \langle P_m, y_n \rangle = \langle P_m, y_0 \rangle$. Therefor there exists a sub series $\{y_{nk}\} \subset \{y_n\}$ such that $\|y_{nk} - y_0\|_p \to 0 (k \to 0)$. From the closeness of the set $(C_E - U_0)$ and $y_{nk} \notin U_0$, we get $y_0 \notin U_0$, which is contradictory to $y_0 \in P_{C_E}(X_0) \subset U_0$. Hence, the optimal equilibrium choice operator $P_{C_E}(X)$ is a upper semi-continuity on T. □

Remark 4. In fact, if the market price vector P_m and consumer's monetary wealth M keep constant, when his material wealth X has a little change in T, his buying behavior also changes a little, which illuminates that rational consumer' buying behavior is Conservative, and does not changes greatly.

Remark 5. The upper semi-continuity of the optimal equilibrium choice operator $P_{C_E}(X)$ has the following economic meaning: when a consumer's optimal choice has been known, you can forecast his future buying behavior through observing the changes of his material wealth.

Remark 6. if a commodity combination has the WKC for a consumer, then he know his optimal choice, which is consistent with the real life.

5 Future Work

This paper explores a consumer's static optimal choice behavior, and gives consumer optimal choice equilibrium characteristic. It is limited that the paper concerns, which only contains equilibrium characteristic and upper semi-continuity of $P_{C_E}(X)$. The uniqueness, existence of the dynamic optimal choice and itself with unsymmetrical information will be discussed in the future study.

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