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Linear Programming for Portfolio Selection Based on Fuzzy Decision-Making Theory

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Abstract In this paper, portfolio selection in crisp and fuzzy cases is studied respectively, and corresponding model and algorithms in both case are proposed. In two models, the risk is taken as the sum of the absolute deviation of the risky assets in stead of covariance, the transaction cost is taken as v-shaped function of the difference between the existing and new portfolio. An efficient way is given to transform an optimal problem with non-linear objective function or non-linear constraint into a linear problem, which alleviate the computational difficulty greatly. The investor's subjective impact is reflected in the model of the fuzzy decision-making environment. Comparison and analysis of the two models is given via a numerical example which has been used in Markowitz's paper [2].

Keywords portfolio selection; fuzzy sets; transaction cost; linear programming; optimization

1 Introduction

Fluctuation in stock market is unpredictable and it is random in nature. This is a difficult task to achieve without planing and evaluating investment alternatives. The portfolio must incorporate what the investor believes to be an acceptable balance between risk and reward. Markowitz's mean-variance model of portfolio selection [1, 2] is one of the best known models in finance and unanimously recognized to contribute in the development of modern portfolio theory. It explores how risk-averse investors can construct optimal portfolio assets taking into consideration the trade-off between expected returns and market risk.

Portfolio selection issue continuously gaining an interest among scholars[3, 4, 14, 11]. Since the computational difficulty of covariance, Markowitz idea on the mean-variance approach then being expended by many researchers such as Sharpe, Mossin, and Lintner. The modern portfolio theory then evolved to Capital Asset Pricing Theory[13] when risk free rate asset was included into the portfolio and then evolved to Arbitrage Pricing Theory in which the computation was largely reduced. Konno&Yamazaki proposed absolutely mean-variance deviation as risk function from another perspective to reduce the model and got efficient result[3]. Furthermore, the Markowitz model is too basic from practical point of view and ignores many constraints faced by real-world investors: trading limitations, size of portfolio, transaction costs, etc[4, 5, 12, 10]. Investment strategies may be theoretically very profitable before taking into account transaction costs and taxation

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issue, but the situation can become worse (such as inefficient portfolio) and completely different when these last constraints are incorporated. Any realistic investment portfolio selection must support transaction costs among other practical limitations.

Several papers dealt with the problem using both quantitative and qualitative analysis methods. One of the hot research topics in this area is the use of fuzzy set theory. Fuzzy set theory[6] is a powerful tool used to describe an uncertain environment with vagueness, ambiguity or some other type of fuzziness, which appears in many aspects of financial markets. Studies by Tanaka et. al. [8],[9],[7], Wang et al. [15], Bilbao-Terol et al. [16], Vercher et al. [17], Lin & Liu, [18] and Li & Xu, [19] show that the fuzzy approach also applicable in portfolio selection.

In order to be easily application, this paper managed to propose a model in which considering transaction cost for avoiding inefficient portfolio firstly, using absolute deviation instead of variance secondly and lastly formulating the nonlinear programming to linear programming.

2 Portfolio selection model under crisp case

Suppose that an investor chooses x_i, the proportion invested in asset i, $1 \leq i \leq n$ for n assets. The constraints are $\sum\limits_{i=1}^{n} x_i = 1$ and $x_i \geq 0, i = 1,2,\cdots n$. The return R_i for the ith asset, $1 \leq i \leq n$, is a random variable, with expected return $r_i = E(R_i)$. Let $R = (R_1, R_2, \cdots, R_n)^T, x = (x_1, x_2, \cdots, x_n)^T$ and $r = (r_1, r_2, \cdots, r_n)^T$. In this paper the transition cost for the ith asset c_i employs v-shape function, that is

$$c_i = k_i|x_i - x_i^0|, \quad i = 1,2,\cdots,n \tag{1}$$

where $x = (x_1^0, x_2^0, \cdots, x_n^0)^T$ is a given assets and $k_i \geq 0$ the transition cost for the unit of ith asset.

So the total transition cost is described

$$\sum_{i=1}^{n} c_i = \sum_{i=1}^{n} k_i|x_i - x_i^0| \tag{2}$$

and the total return is

$$R(x) = E\left[\sum_{i=1}^{n} R_i x_i\right] - \sum_{i=1}^{n} k_i|x_i - x_i^0| = \sum_{i=1}^{n} r_i x_i - \sum_{i=1}^{n} k_i|x_i - x_i^0| \tag{3}$$

thus the total risk can be given as follow:

$$V(x) = \sum_{i=1}^{n} E|(R_i - E(R_i))x_i| = \sum_{i=1}^{n} d_i x_i \tag{4}$$

where $d_i = E|(R_i - E(R_i))x_i|$.

In general, investors expect maximizing returns and minimizing risk at the meantime.

It can be formulated mathematically as two-objective Programming Model

$$\begin{array}{l} \max\ R(x) = \sum_{i=1}^{n} r_i x_i - \sum_{i=1}^{n} k_i |x_i - x_i^0| \\ \min\ V(x) = \sum_{i=1}^{n} d_i x_i \\ s.t.\ \sum_{i=1}^{n} x_i = 1, x_i \geq\ 0, i = 1, \cdots, n \end{array} \tag{5}$$

Weighted sum approach to simplify the multi-objective problems, we get the following parametric programming

$$\begin{array}{l} \max\ (1-\lambda)\left(\sum_{i=1}^{n} r_i x_i - \sum_{i=1}^{n} k_i |x_i - x_i^0|\right) - \lambda \sum_{i=1}^{n} d_i x_i \\ s.t. \begin{cases} \sum_{i=1}^{n} x_i = 1 \\ x_i \geq\ 0, i = 1, \cdots, n \end{cases} \end{array} \tag{6}$$

where $\lambda \in [0,1]$ is called risk-aversion factor. The greater value λ is, the more awareness of risk aversion.

Theorem 1. *$x^* = (x_1^*, x_2^*, \cdots, x_n^*)$ is an optimal solution of the model (6) if and only if there exist $(y_1^*, y_2^*, \cdots, y_n^*)$ such that $(x_1^*, x_2^*, \cdots, x_n^*; y_1^*, y_2^*, \cdots, y_n^*)$ is an optimal solution of the following programming:*

$$\begin{array}{l} \max\ (1-\lambda)\left(\sum_{i=1}^{n} r_i x_i - \sum_{i=1}^{n} k_i y_i\right) - \lambda \sum_{i=1}^{n} d_i x_i \\ s.t. \begin{cases} y_i + x_i - x_i^0 \geq\ 0 \\ y_i - x_i + x_i^0 \geq\ 0 \\ \sum_{i=1}^{n} x_i = 1 \\ x_i \geq\ 0, i = 1, \cdots, n \end{cases} \end{array} \tag{7}$$

Proof. Suppose that $x^* = (x_1^*, x_2^*, \cdots, x_n^*)$ is an optimal solution of the model (6), let $y_i^* = |x_i^* - x_i^0|$. It is obvious that $(x_1^*, x_2^*, \cdots, x_n^*; y_1^*, y_2^*, \cdots, y_n^*)$ is a feasible solution of (7). It need prove that $(x_1^*, x_2^*, \cdots, x_n^*; y_1^*, y_2^*, \cdots, y_n^*)$ is an optimal solution of (7).

Let $x = (x_1, x_2, \cdots, x_n; y_1, y_2, \cdots, y_n)$ is any feasible solution of (7), then $x = (x_1, x_2, \cdots, x_n)$ is a feasible solution of (6). Since $x^* = (x_1^*, x_2^*, \cdots, x_n^*)$ is an optimal solution of the model (6), we have

$$\begin{array}{rl} & (1-\lambda)\left(\sum_{i=1}^{n} r_i x_i^* - \sum_{i=1}^{n} k_i |x_i^* - x_i^0|\right) - \lambda \sum_{i=1}^{n} d_i x_i^* \\ \geq & (1-\lambda)\left(\sum_{i=1}^{n} r_i x_i - \sum_{i=1}^{n} k_i |x_i - x_i^0|\right) - \lambda \sum_{i=1}^{n} d_i x_i \\ \geq & (1-\lambda)\left(\sum_{i=1}^{n} r_i x_i - \sum_{i=1}^{n} k_i y_i\right) - \lambda \sum_{i=1}^{n} d_i x_i \end{array} \tag{8}$$

thus

$$\begin{aligned} & (1-\lambda)\left(\sum_{i=1}^{n} r_i x_i^* - \sum_{i=1}^{n} k_i y_i^*\right) - \lambda \sum_{i=1}^{n} d_i x_i^* \\ \geq\ & (1-\lambda)\left(\sum_{i=1}^{n} r_i x_i - \sum_{i=1}^{n} k_i y_i\right) - \lambda \sum_{i=1}^{n} d_i x_i \end{aligned} \tag{9}$$

that is, $(x_1^*, x_2^*, \cdots, x_n^*; y_1^*, y_2^*, \cdots, y_n^*)$ is an optimal solution of (7).

On the contrary, let $(x_1^*, x_2^*, \cdots, x_n^*; y_1^*, y_2^*, \cdots, y_n^*)$ be an optimal solution of (7). We prove that $x^* = (x_1^*, x_2^*, \cdots, x_n^*)$ is an optimal solution of the model (6). Obviously, $x^* = (x_1^*, x_2^*, \cdots, x_n^*)$ is an feasible solution of the model (6) and $y_i^* \geq |x_i^* - x_i^0|$. If $x^* = (x_1^*, x_2^*, \cdots, x_n^*)$ is not optimal solution of the model (6), there exists a feasible solution $x = (x_1, x_2, \cdots, x_n)$ of (6) such that

$$\begin{aligned} & (1-\lambda)\left(\sum_{i=1}^{n} r_i x_i - \sum_{i=1}^{n} k_i |x_i - x_i^0|\right) - \lambda \sum_{i=1}^{n} d_i x_i \\ >\ & (1-\lambda)\left(\sum_{i=1}^{n} r_i x_i^* - \sum_{i=1}^{n} k_i |x_i^* - x_i^0|\right) - \lambda \sum_{i=1}^{n} d_i x_i^* \end{aligned} \tag{10}$$

let $y_i = |x_i - x_i^0|, i = 1, 2, \cdots, n$, then $x = (x_1, x_2, \cdots, x_n; y_1, y_2, \cdots, y_n)$ is a feasible solution of (7) and

$$\begin{aligned} & (1-\lambda)\left(\sum_{i=1}^{n} r_i x_i - \sum_{i=1}^{n} k_i y_i\right) - \lambda \sum_{i=1}^{n} d_i x_i \\ =\ & (1-\lambda)\left(\sum_{i=1}^{n} r_i x_i - \sum_{i=1}^{n} k_i |x_i - x_i^0|\right) - \lambda \sum_{i=1}^{n} d_i x_i \\ >\ & (1-\lambda)\left(\sum_{i=1}^{n} r_i x_i^* - \sum_{i=1}^{n} k_i |x_i^* - x_i^0|\right) - \lambda \sum_{i=1}^{n} d_i x_i^* \\ \geq\ & (1-\lambda)\left(\sum_{i=1}^{n} r_i x_i^* - \sum_{i=1}^{n} k_i y_i^*\right) - \lambda \sum_{i=1}^{n} d_i x_i^* \end{aligned} \tag{11}$$

□

From the above discussion, we can see that it can make the complex portfolio selection simplify if properly constructing the risk function and simplifying the portfolio selection model.

3 Portfolio selection model under fuzzy case

In an investment, the knowledge and experience of experts are very important in an investor's decision-making. Due to complexity and un-prediction in financial markets, it is difficult to give the precise expected value about the risk and return, thus it can be taken risk and return for granted as two fuzzy objectives. Since an investor can accept the return greater than some level and accept the risk less than some level of that, the membership function of two fuzzy objectives μ_{max} and μ_{min} can be given by

$$\mu_{max}(x) = \begin{cases} 0, & \sum_{i=1}^{n} r_i x_i - \sum_{i=1}^{n} k_i |x_i - x_i^0| \leq S_0 \\ \dfrac{\sum_{i=1}^{n} r_i x_i - \sum_{i=1}^{n} k_i |x_i - x_i^0| - S_0}{S_1 - S_0}, & S_0 \leq \sum_{i=1}^{n} r_i x_i - \sum_{i=1}^{n} k_i |x_i - x_i^0| \leq S_1 \\ 1, & \sum_{i=1}^{n} r_i x_i - \sum_{i=1}^{n} k_i |x_i - x_i^0| \geq S_1 \end{cases} \tag{12}$$

$$\mu_{min}(x) = \begin{cases} 0, & \sum_{i=1}^{n} d_i x_i \geq T_0 \\ \dfrac{T_0 - \sum_{i=1}^{n} d_i x_i}{T_0 - T_1}, & T_1 \leq \sum_{i=1}^{n} d_i x_i \leq T_0 \\ 1, & \sum_{i=1}^{n} d_i x_i \leq T_1 \end{cases} \tag{13}$$

where S_0, S_1, T_0, T_1 are given by the investor.

By introducing the variable μ and from the theory of fuzzy set and fuzzy programming, we can construct the following programming

$$\max \ \mu$$

$$s.t. \begin{cases} \dfrac{\sum_{i=1}^{n} r_i x_i - \sum_{i=1}^{n} k_i |x_i - x_i^0| - S_0}{S_1 - S_0} \geq \mu \\ \dfrac{T_0 - \sum_{i=1}^{n} d_i x_i}{T_0 - T_1} \geq \mu \\ \sum_{i=1}^{n} x_i = 1 \\ x_i \geq 0, i = 1, \cdots, n \end{cases} \tag{14}$$

Theorem 2. *x^* is an optimal solution of (14) if and only if there exists y^* such that (x^*, y^*) is an optimal solution of the following programming*

$$\max \ \mu$$

$$s.t. \begin{cases} \dfrac{\sum_{i=1}^{n} r_i x_i - \sum_{i=1}^{n} k_i y_i - S_0}{S_1 - S_0} \geq \mu \\ \dfrac{T_0 - \sum_{i=1}^{n} d_i x_i}{T_0 - T_1} \geq \mu \\ y_i + x_i - x_i^0 \geq 0 \\ y_i - x_i + x_i^0 \geq 0 \\ \sum_{i=1}^{n} x_i = 1 \\ x_i \geq 0, i = 1, \cdots, n \end{cases} \tag{15}$$

Proof. It is similar to the proof the theorem 1. We omit it here. □

4 Numerical example and Conclusion

In this section, we will give a numerical example to illustrate the proposed portfolio selection model (7) and (14). We suppose that the investor considers the stock portfolio selection in Markowitz's paper [2] where the data as shown in the table 1.

Table 1: The return of American Tabacoo, AT&T, United Stats, General Motors, Atchison&Topeka&Santa Fe, Coca-Cola, Borden, Firestone and Sharon Steel(1937-1954)

Year	# 1	# 2	# 3	# 4	# 5	# 6	# 7	# 8	# 9
1937	-0.305	-0.173	-0.318	-0.477	-0.457	-0.065	-0.319	-0.4 ąą	-0.435
1938	0.513	0.098	0.285	0.714	0.107	0.238	0.076	0.336	0.238
1939	0.055	0.2	-0.047	0.165	-0.424	-0.078	0.381	-0.093	-0.295
1940	-0.126	0.03	0.104	-0.043	-0.189	-0.077	-0.051	-0.09	-0.036
1941	-0.28	-0.183	-0.171	-0.277	0.637	-0.187	0.087	-0.194	-0.24
1942	-0.003	0.067	-0.039	0.476	0.865	0.156	0.262	1.113	0.126
1943	0.428	0.3	0.149	0.225	0.313	0.351	0.341	0.58	0.639
1944	0.192	0.103	0.26	0.29	0.637	0.233	0.227	0.473	0.282
1945	0.446	0.216	0.419	0.216	0.373	0.349	0.352	0.229	0.578
1946	-0.088	-0.046	-0.078	-0.272	-0.037	-0.209	0.153	-0.126	0.289
1947	-0.127	-0.071	0.169	0.144	0.026	0.355	-0.099	0.009	0.184
1948	-0.015	0.056	-0.035	0.107	0.153	-0.231	0.038	0	0.114
1949	0.305	0.038	0.133	0.321	0.067	0.246	0.273	0.223	-0.222
1950	-0.096	0.089	0.732	0.305	0.579	-0.248	0.091	0.65	0.327
1951	0.016	0.09	0.021	0.195	0.04	-0.064	0.054	-0.131	0.333
1952	0.128	0.083	0.131	0.39	0.434	0.079	0.109	0.175	0.062
1953	-0.01	0.035	0.006	-0.072	-0.027	0.067	0.21	-0.084	-0.048
1954	0.154	0.176	0.908	0.715	0.469	0.077	0.112	0.756	0.185

All computations were carried out on a WINDOWS PC using the LINDO solver. According to the awareness of the investor's risk aversion, we can get the corresponding invest strategies by solving to the model (7). The table 2 shows the obtained part of the results.

Table 2: Part results to model 7

λ	$(x_1, x_2, \cdots, x_9)$	return	risk
0.0	(0, 0, 0, 0, 1, 0, 0, 0, 0)	0.193	0.302
0.3	(0, 0, 0, 1, 0, 0, 0, 0, 0)	0.168	0.235
0.5	(0, 0, 0, 0, 0, 0, 1, 0, 0)	0.123	0.131
1.0	(0, 1, 0, 0, 0, 0, 0, 0, 0)	0.057	0.089

According to the investor's aspiration and the given value S_0, S_1, T_0, T_1, we can get the corresponding strategies by solving to the model (14) as shown in table 3.

Regarding the expected excess return and the tracking error as two objective functions, we have proposed a bi-objective programming model for the index tracking portfolio selection problem. Furthermore, investors' vague aspiration levels for the excess return and

Table 3: Part solutions to model 14

S_0	S_1	T_0	T_1	$(x_1, x_2, \cdots, x_9)$	μ
0.0878	0.1054	0.502	0.202	(0, 0, 0, 0, 0.4152, 0, 0.5848, 0, 0)	1
0.0988	0.20	0.402	0.282	(0, 0, 0, 0.1209, 0.8791, 0, 0, 0, 0)	0.90087

the tracking error are considered as fuzzy numbers. Based on fuzzy decision theory, we have proposed a fuzzy index tracking portfolio selection model. An example is given to illustrate that the proposed fuzzy index tracking portfolio selection model. The computation results show that the proposed model can generate a favorite portfolio strategy according to the investor's satisfactory degree.

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