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Stochastic Cash Management Problem with Double Exponential Jump Diffusion Processes

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Abstract In this paper, we investigate the effect of a sharp cash level fluctuation resulting from the inflow and outflow of a large amount of cash and how the cash balance is managed. We describe the cash level evolution as stochastic jump-diffusion process with double exponential distributed jump size, and formulate a cash management model for minimizing the sum of the transaction and holding-penalty costs. This model can be formulated as an impulse control model, and we derived the cost function under the assumption that a band-type policy exists. Moreover, we discuss the effect of such a fluctuation on the optimal policy though some numerical examples. Consequently, we show the cost function explicit and clarify that the size of sharp fluctuation has strong implications for the optimal policy.

Keywords Cash Balance Management; Jump Diffusion; Impulse Control

1 Introduction

A company trying to implement smooth business activity needs to hold some cash to prepare for an unscheduled settlement risk. In this paper, we investigate the effect of the cash demand risk which means sudden increase or decrease in a large amount of cash and how the cash balance is managed. A large amount of cash demand arises due to natural or human-caused disaster, for example, earthquake, tsunami, terrorism and financial crash. On the contrary, unpredictable inflow of a large amount of cash occurs when an item will be extremely popular or the debt will be refunded. Such a risk is called "exogenous risk". Moreover, there is intrinsic uncertainty in cash demand risk which is called "inherent risk". One of the typical occasions is the time lag between the necessary expenses for production and income from future sales. Under these uncertainties, the objective of the management is to find an optimal cash balance to minimize total cost consisting of transaction and holding-penalty costs for the cash.

Cash management problem has been studied by various researchers by extended the continuous-review stochastic inventory models (e.x. Constantinides and Richard [6], Baccarin [1] and Baccarin [2]). The inherent risk has been described as diffusion process in these papers. The exogenous risk has been described as compound Poisson demand (e.x.

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Federgruen and Schechner [8], Zipkin [12]) in inventory literature or jump diffusion process in finance literature (e.x. Kou and Wang [9], Sepp [11]). The most related research to our study is Bensoussan, Liu and Sethi [4]. They studied an inventory model with a fixed ordering cost and a general demand process that consists of a compound Poisson demand and a diffusion process. They showed that an (s, S) policy is optimal by using a quasi-variational inequality (QVI) when the demand is a mixture of a diffusion process and a compound Poisson process with exponentially distributed jumps size. Following research of Bensoussan, Liu and Sethi [4], Benkherouf and Bensoussan [3] showed the general case where the demand is a combination of a diffusion and a general compound Poisson process with nonnegative jump size. In our study, we, instead, consider double exponentially distributed jumps size to deal with the negative jump size, since there are not only the outflows of a large amount of cash, but also are the inflows of a large amount of cash.

In this paper, we derive an explicit formula for the solution of QVI by assuming that an optimal policy is obtained as a band policy. The band policy is as follows: when the cash level falls to d (rise to u), then it is adjusted up to level D (down to U), $d < D < U < u$. In order for clarity the effects of the exogenous risk on the optimal policy, we present some numerical examples in limited policy having only the two thresholds d and D.

The rest of this paper is organized as follows: In the next section, we construct a mathematical model to describe the cash management problem and use an impulse control to solve it. In the third section, we present numerical studies and the final section concludes the research.

2 The Model

Consider a manager who manages the cash balance of a firm faces with the two uncertainties of the demand: an inherent risk and an exogenous risk. Let Z_t be the cumulative demand in the interval $[0,t]$ and is given by

$$Z_t = \mu t + \sigma W_t + \sum_{i=1}^{N_t} Y_i, \tag{1}$$

where μ is drift, σ is the volatility and W_t is a standard Brownian motion with $W_0 = 0$. N_t is a Poisson process with rate $\lambda \geq 0$, and Y_i, $i = 1, 2, \cdots$, is a sequence of i.i.d. random variables having distribution density $m(\cdot)$. We assume that W_t, N_t and Y_i are all independent. According to the model, the cumulate demand consists of three factors: a deterministic trend of demand μ, inherent risk described by the Brownian motion, and exogeneous risk captured by the Poisson-arrival jump part. In our research, we assume that Y_i has a double exponential distribution and its density function is given by

$$m(y) = p\eta_1 e^{-\eta_1 y} 1_{\{y \geq 0\}} + q\eta_2 e^{\eta_2 y} 1_{\{y < 0\}}, \tag{2}$$

where $\eta_1, \eta_2 > 0$ and $p, q \geq 0$ such that $p + q = 1$. The first and second moments of Y are given by $E[Y] = p/\eta_1 - q/\eta_2 \equiv \mu_m$ and $E[Y^2] = 2(p/\eta_1^2 + q/\eta_2^2) \equiv \hat{\mu_m}$, respectively.

A cash management policy consists of a sequence (τ_i, ξ_i), $i = 1, 2, \cdots$, where τ_i represents the ith time of adjusting and ξ_i represents the quantity adjusted at time τ_i, where $\tau_1 < \tau_2 < \cdots$. If the policy $v = \{(\tau_i, \xi_i), i = 1, 2, \cdots\}$ is adopted, then the cash level at time t evolves as

$$X_t^v = x - Z_t + \sum_{i=1}^{\infty} I_{\{\tau_i < t\}} \xi_i, \tag{3}$$

where $X_0^v = x$ is an initial cash level at time $t = 0$.

When the cash level changes from x to $x + \xi$, the transition costs occur. We denote a fixed cost by K_1 (K_2) and proportional cost by k_1 (k_2) if the manager increases (decreases) the cash. Transaction costs function is the sum of fixed and proportional costs of form

$$T(\xi_i) = \begin{cases} K_1 + k_1 \xi_i, & \text{if } \xi_i \geq 0, \\ K_2 + k_2 |\xi_i|, & \text{if } \xi_i < 0. \end{cases} \tag{4}$$

We assume that holding and penalty costs function $C(x)$ to be quadratic form as

$$C(x) = \begin{cases} h_1 x + h_2 x^2, & \text{if } x \geq 0, \\ -p_1 x + p_2 x^2, & \text{if } x < 0. \end{cases} \tag{5}$$

Baccarin [1] treated the problem with the quadratic holding-penalty function in (5) for no-jump case.

Given initial cash level x and control v, the associated total expected cost is given by

$$J_x(v) = E_x^v \left[\int_0^{\infty} C(X_t^v) e^{-\alpha t} dt + \sum_{i=1}^{\infty} T(\xi_i) e^{-\alpha \tau_i} \mid X_0^v = x \right] \tag{6}$$

where $\alpha > 0$ is discount rate. Then, our objective is to find a policy v^* to minimize the expected total cost with the value function

$$\Phi(x) = \inf_{v \in \mathscr{V}} J_x(v). \tag{7}$$

Next we consider the formulating the problem given by (7) as a Quasi-Variational inequality problem. We introduce the following operators:

$$(M\phi)(x) = \inf_{\xi} \{ T(\xi) + \phi(x + \xi) \}, \tag{8}$$

$$\begin{aligned} (A\phi)(x) = & -\frac{1}{2}\sigma^2 \phi''(x) + \mu \phi'(x) + \alpha \phi(x) \\ & -\lambda \int_{-\infty}^{\infty} (\phi(x-y) - \phi(x) + y\phi'(x)) m(y) dy. \end{aligned} \tag{9}$$

Then, the optimal expected cost for the cash management model is given as a solution of the following (QVI) problem.

Definition 2.1.
A function ϕ satisfies the quasi-variational inequalities for the problem (7) if, for every x,

$$A\phi \leq C, \tag{10}$$

$$\phi \leq M\phi, \tag{11}$$

$$(A\phi - C)(\phi - M\phi) = 0. \tag{12}$$

Davis, Guo and Wu [7] showed regularity properties of the vaule function for an infinite-horizon discounted cost impulse control problem, where the underlying controlled process is a multidimensional jump diffusion. Thus, if there is a sufficiently regular solution of (10) - (12), the solution is the value function.

3 Candidate Function for Value Function and the Optimal Policy

We expect that an optimal policy is determined by parameters (d, D, U, u) where $-\infty < d < D < U < u < \infty$. When the cash level is x, we transact x to level $\delta(x)$ where

$$\delta(x) = \begin{cases} D, & \text{if } x \leq d, \\ x, & \text{if } d \leq x \leq u, \\ U, & \text{if } u \leq x. \end{cases} \tag{13}$$

Then, the function ϕ would satisfies $A\phi = C$ for $d \leq x \leq u$, that is,

$$-\frac{1}{2}\sigma^2\phi^{''}(x) + (\mu - \lambda\mu_m)\phi^{'}(x) + (\lambda + \alpha)\phi(x) - \lambda\int_{-\infty}^{\infty}\phi(x-y)m(y)dy - C(x) = 0 \tag{14}$$

and

$$\phi(x) = \begin{cases} \phi(D) + K_1 + k_1(D - x), & \text{for } x \leq d, \\ \phi(U) + K_2 + k_2(x - U), & \text{for } x \geq u. \end{cases} \tag{15}$$

Moreover, if ϕ is differentiable, neccessary conditions for optimality of the actions in (d,u) are

$$\phi^{'}(D) = -k_1, \quad \phi^{'}(U) = k_2. \tag{16}$$

And, continuity of the derivative of the value function requires

$$\phi^{'}(d) = -k_1, \quad \phi^{'}(u) = k_2. \tag{17}$$

In order to find the general solution of the equation (14), we need an function $G(\cdot)$ such as

$$G(\theta) = -(\mu - \lambda\mu_m)\theta + \frac{1}{2}\sigma^2\theta^2 + \lambda\left(\frac{p\eta_1}{\eta_1 + \theta} + \frac{q\eta_2}{\eta_2 - \theta} - 1\right). \tag{18}$$

Kou and Wang (2003) showed that the four roots of an equation $G(\theta)=\alpha$ for any $\alpha>0$ are real numbers. Denote them by β_1, β_2, β_3, β_4. These roots satisfy

$$-\infty<\beta_4<-\eta_1<\beta_3<0<\beta_2<\eta_2<\beta_1<\infty. \tag{19}$$

Proposition 3.1.
Suppose that there exist four parameters d, D, U and u. Then, the function ϕ satisfying the equations (14) and (15) are given by

$$\phi(x)=\begin{cases}\phi(d)+k_1(d-x), & \text{if } x\le d,\\ (A_1+A_5)e^{\beta_1 x}+(A_2+A_6)e^{\beta_2 x}+A_7e^{\beta_3 x}+A_8e^{\beta_4 x}+\psi_1(x), & \text{if } d\le x\le 0,\\ (A_3+A_7)e^{\beta_3 x}+(A_4+A_8)e^{\beta_4 x}+A_5e^{\beta_1 x}+A_6e^{\beta_2 x}+\psi_2(x), & \text{if } 0\le x\le u,\\ \phi(u)+k_2(x-u), & \text{if } u\le x,\end{cases} \tag{20}$$

where

$$\begin{aligned}\psi(x) &= \begin{pmatrix}\frac{p_2}{\alpha}\\ \frac{h_2}{\alpha}\end{pmatrix}x^2+\begin{pmatrix}-\frac{1}{\alpha^2}(\alpha p_1+2\mu p_2)\\ \frac{1}{\alpha^2}(\alpha h_1-2\mu h_2)\end{pmatrix}x\\ &\qquad +\begin{pmatrix}\frac{1}{\alpha^2}p_2(\sigma^2+\lambda\hat{\mu_m})+\frac{1}{\alpha^3}\mu(\alpha p_1+2\mu p_2)\\ \frac{1}{\alpha^2}h_2(\sigma^2+\lambda\hat{\mu_m})-\frac{1}{\alpha^3}\mu(\alpha h_1-2\mu h_2)\end{pmatrix}\\ &\equiv \boldsymbol{a}x^2+\boldsymbol{b}x+\boldsymbol{c}.\end{aligned} \tag{21}$$

The constants A_j, $j=1,\cdots,4$, are solutions of the equations

$$\begin{pmatrix}1 & 1 & -1 & -1\\ \beta_1 & \beta_2 & -\beta_3 & -\beta_4\\ \frac{1}{\eta_1+\beta_1} & \frac{1}{\eta_1+\beta_2} & -\frac{1}{\eta_1+\beta_3} & -\frac{1}{\eta_1+\beta_4}\\ \frac{1}{\eta_2-\beta_1} & \frac{1}{\eta_2-\beta_2} & -\frac{1}{\eta_2-\beta_3} & -\frac{1}{\eta_2-\beta_4}\end{pmatrix}\begin{pmatrix}A_1\\ A_2\\ A_3\\ A_4\end{pmatrix} = \begin{pmatrix}c_2-c_1\\ b_2-b_1\\ \frac{1}{\eta_1^3}\{2(a_2-a_1)-\eta_1(b_2-b_1)+\eta_1^2(c_2-c_1)\}\\ \frac{1}{\eta_2^3}\{2(a_2-a_1)+\eta_2(b_2-b_1)+\eta_2^2(c_2-c_1)\}\end{pmatrix}, \tag{22}$$

*where a_i, b_i, c_i $(i=1,2)$ are elements of vactor **a**, **b** and **c**, respectively. Furthermore, the policy parameters d, D, U, u and constants A_j, $j=5,\cdots,8$, are given by the solutions of* *the equations (16), (17) and equations*

$$\begin{pmatrix} \frac{e^{\beta_1 d}}{\eta_1+\beta_1} & \frac{e^{\beta_2 d}}{\eta_1+\beta_2} & \frac{e^{\beta_3 d}}{\eta_1+\beta_3} & \frac{e^{\beta_4 d}}{\eta_1+\beta_4} \\ e^{\beta_1 d} & e^{\beta_2 d} & e^{\beta_3 d} & e^{\beta_4 d} \\ e^{\beta_1 u} & e^{\beta_2 u} & e^{\beta_3 u} & e^{\beta_4 u} \\ \frac{e^{\beta_1 u}}{\eta_2-\beta_1} & \frac{e^{\beta_2 u}}{\eta_2-\beta_2} & \frac{e^{\beta_3 u}}{\eta_2-\beta_3} & \frac{e^{\beta_4 u}}{\eta_2-\beta_4} \end{pmatrix} \begin{pmatrix} A_5 \\ A_6 \\ A_7 \\ A_8 \end{pmatrix}$$
$$= \begin{pmatrix} -\frac{A_1}{\eta_1+\beta_1} e^{\beta_1 d} - \frac{A_2}{\eta_2+\beta_2} e^{\beta_2 d} + \frac{1}{\eta_1}\{\phi(D)+K_1+k_1(D-d+\frac{1}{\eta_1})+\chi_1\} \\ -A_1 e^{\beta_1 d} - A_2 e^{\beta_2 d} - \psi_1(d)+\phi(D)+K_1+k_1(D-d) \\ -A_3 e^{\beta_3 u} - A_4 e^{\beta_4 u} - \psi_2(u)+\phi(U)+K_2+k_2(u-U) \\ -\frac{A_3}{\eta_2-\beta_3} e^{\beta_3 u} - \frac{A_4}{\eta_2-\beta_4} e^{\beta_4 u} + \frac{1}{\eta_2}\{\phi(U)+K_2+k_2(u-U+\frac{1}{\eta_2})+\chi_2\} \end{pmatrix} \tag{23}$$

where

$$\chi_1 = -\frac{1}{\eta_1^2}\{\eta_1^2(a_1 d^2+b_1 d+c_1)-\eta_1(2a_1 d+b_1)+2a_1\}, \tag{24}$$

$$\chi_2 = -\frac{1}{\eta_2^2}\{\eta_2^2(a_2 u^2+b_2 u+c_2)+\eta_2(2a_2 u+b_2)+2a_2\}. \tag{25}$$

We present a special case in which there are no upper levels u and U. Thus, the manager does not adjust the cash level downward when the cash balance is high. Let $u \to \infty$ and set $A_5 = A_6 = 0$, the value function $\phi^D(x)$ is given by

$$\phi^D(x) = \begin{cases} \phi^D(d)+k_1(d-x), & \text{if } x \le d, \\ A_1 e^{\beta_1 x}+A_2 e^{\beta_2 x}+A_7 e^{\beta_3 x}+A_8 e^{\beta_4 x}+\psi_1(x), & \text{if } d \le x \le 0, \\ (A_3+A_7)e^{\beta_3 x}+(A_4+A_8)e^{\beta_4 x}+\psi_2(x), & \text{if } 0 \le x, \end{cases} \tag{26}$$

where A_1, A_2, A_3 and A_4 are solutions of the equations (22). The parameters d and D and the values of A_7 and A_8 can be obtained by the solutions to the following equations;

$$\begin{cases} \phi^D(d) = \phi^D(D) = -k_1, \\ \phi^D(d)-\phi^D(D) = K_1+k_1(D-d). \end{cases} \tag{27}$$

4 Numerical Example

We provide in this section numerical examples to illustrate the effect of exogenous risk on the optimal policy for the two-level case described in equation (26). We set $\mu = 0.1$, $\sigma = 0.2$, $\lambda = 3$, $p = 0.5$, $q = 0.5$, $\eta_1 = 200$, $\eta_2 = 300$, $\alpha = 0.01$, $h_1 = 0.1$, $h_2 = 0.15$, $p_1 = 0.4$, $p_2 = 0.5$, $k_1 = 0.2$, $K_1 = 0.8$. The optimal policy for our model is obtained by solving the equations (27), and the parameters are $d = -0.4646$, $D = 0.5242$. Figure 1 shows the value functions for our model and Baccarin [1]'s model, which has no exogenous risk, as the functions of the cash level x. The form of the value function for our model (Jump case) is unimodal with respect to x, and the expected total cost of our model is less than that of Baccarin [1]'s model. The thresholds for jump case are small in comparison to the one for no-jump case: $\tilde{d} = -0.6557$, $\tilde{D} = 0.8709$.

The sensitivity analysis for the thresholds d and D are performed by varying different parameters λ, η_1, η_2 and p, and the results are given in Figure 2 to 7. Figure 2 and Figure 3 show that the thresholds d and D decrease with λ, which the rate of the Poisson process. Figure 4 and Figure 5 show that d and D are decreasing in p, which the probability of the downward jump for cash level. Figure 6 and Figure 7 indicate that the thresholds are increasing in η_1 and η_2. Thus, the larger the mean of the jump size, the smaller the value of the thresholds d and D. Moreover, we can see that the values η_1 and η_2 have a large effect on the optimal policy.

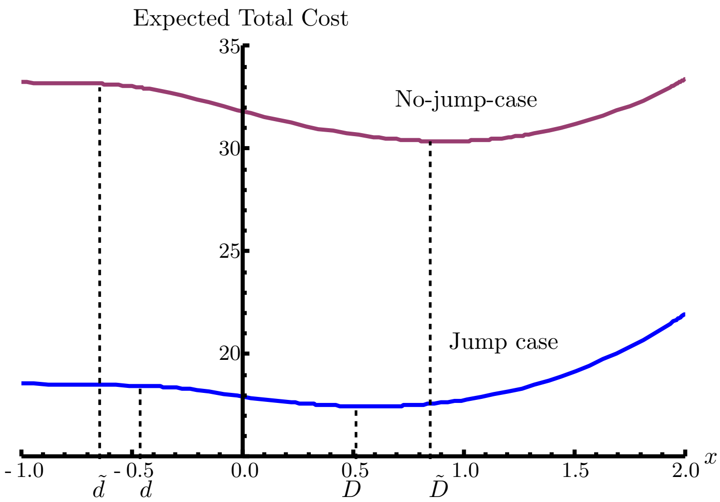

Figure 1: Form of the vaule function

5 Concluding Remarks

In this paper we have considered the cash management model in which the cash level suddenly increase or decrease in a large amount. It has shown that such cash management model can be formulated as a quasi-variational inequality problem. We have derived an explicit cost function for this model under the assumption that an optimal policy obtained as a band policy. Then, we have investigated the effect of the exogenous risk on the optimal policy. As a result, the size of the demand for exogenous risk has strong implications for optimal policy.

In the future, we will show that the obtained value function satisfies the quasi-variational inequality and that the existence of the policy parameters. In addition, we would like to apply our model to practical problem such as the management of an automated teller machine (ATM).

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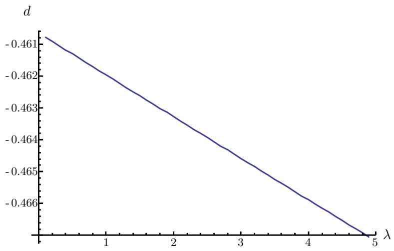

Figure 2: Optial threshold for d with respect to λ.

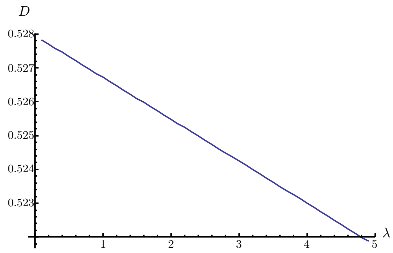

Figure 3: Optimal threshold for D with respect to λ.

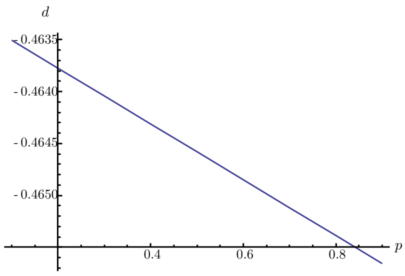

Figure 4: Optimal threshold for d with respect to p.

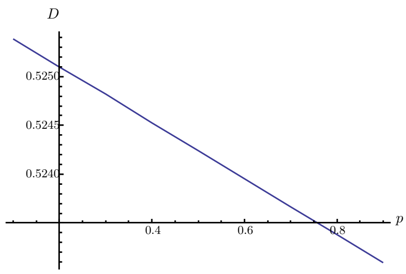

Figure 5: Optimal threshold for D with respect to p.

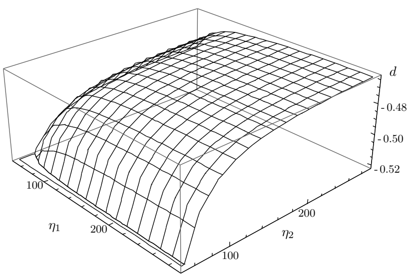

Figure 6: Optimal threshold for d with respect to η_1 and η_2.

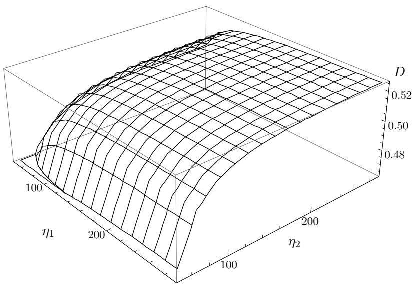

Figure 7: Optimal threshold for D with respect to η_1 and η_2.