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The Multi-Level Economic Lot-Sizing Game

Gai-Di Li[1] Dong-Lei Du[2] Da-Chuan Xu[1,3]
Ru-Yao Zhang[4]

[1]Department of Applied Mathematics, Beijing University of Technology
100 Pingleyuan, Chaoyang District, Beijing 100124, P.R. China. Email: {ligd; xudc}@bjut.edu.cn.
[2]Faculty of Business Administration, University of New Brunswick
Fredericton, NB Canada E3B 5A3. Email: ddu@unb.ca.
[3]Corresponding author.
[4]School of Mathematical Sciences, Fudan University
Shanghai 200433, P.R. China. Email: miranda_cheung@yahoo.cn.

Abstract We present a cost-sharing method that is cross-monotonic, competitive, and approximate cost recovery, for the multi-level economic lot-sizing game, under a mild condition. This result extends that of the recent 1-level economic lot-sizing game of Xu and Yang (2009).

Keywords Multi-level economic lot-sizing game, cross-monotonic, competitive, approximate cost recovery.

1 Introduction

Multi-level economic lot-sizing models have been extensively investigated under considerably different multi-level structure assumptions in the literature (c.f., [6, 7, 17]). The model we are interested in is similar to those in [7, 17]. However, the multi-level system of this paper will be analyzed as an economic lot-sizing system with backlogging demand permitted (i.e., unfulfilled demands can be met later). In this multi-level system, the manufacturing of the final product requires several different processes. Each process is assumed to take place in exactly one level or facility. Adopting the convention that finished goods belong to level 0, intermediate products are then numbered from 1 to k, starting from level 0 and all the way to the lowest level k. The objective is to determine the production schedule in each level such that all demands are fulfilled and total cost is minimized.

To formally describe the problem, we need the following notations.

T :	total number of periods;
d_t :	the amount of demand at period t;
$x_{i,t}$:	the processed quantity of level i at period t;
$I_{i,t}$:	the inventory quantity of level i at period t;
$r_{i,t}$:	the amount of backlogged demand for level i at period t;
$h_{i,t}$:	the unit holding cost of level i at period t;
$g_{i,t}$:	the unit backlogging cost of level i at period t;
$p_{i,t}$:	the unit producing cost of level i at period t;
$f_{i,t}$:	the setup cost of level i at period t.

Then the multi-level economic lot-sizing problem (MLELSP) can be formulated as the following mathematical programming:

$$\begin{array}{rll} C(d) := \min & \sum_{t=1}^{T}\sum_{i=1}^{k}\{f_{i,t}\delta(x_{i,t}) + p_{i,t}x_{i,t} + h_{i,t}I_{i,t} + g_{i,t}r_{i,t}\} & \\ \text{s.t.} & I_{i,0} = I_{i,T} = r_{i,0} = r_{i,T} = 0, & \forall i = 1,\cdots,k, \\ & x_{i,t} + I_{i,t-1} - r_{i,t-1} = x_{i-1,t} + I_{i,t} - r_{i,t}, & \forall t = 1,\cdots,T; i = 2,\cdots,k, \\ & x_{1,t} + I_{1,t-1} - r_{1,t-1} = d_t + I_{1,t} - r_{1,t}, & \forall t = 1,\cdots,T, \\ & x_{i,t} \geq 0, I_{i,t} \geq 0, & \forall t = 1,2,\cdots,T; i = 1,\cdots,k, \end{array}$$

where $d = (d_1, d_2, \cdots, d_T)^T$ is the demand vector and

$$\delta(x) = \begin{cases} 1, & \text{if } x > 0; \\ 0, & \text{otherwise.} \end{cases}$$

In this paper, we consider the cooperative game associated with the above MLELSP. We consider the situation with multiple manufacturers supplying the same product. In a decentralized system, each manufacturer would solve a MLELSP. However, by exploiting economies of scale, the manufacturers may find it beneficial to form coalitions and make joint production, leading to the *multi-level economic lot-sizing game* (MLELSG), a cooperative game with the players being manufacturers. In cooperative game theory, the central problem is to develop cost allocations that are advantageous to all manufacturers, i.e., no manufacturer(s) have an incentive to secede.

Formally, consider a set of manufacturers $N = \{1, 2, \cdots, n\}$, who produce the same goods to meet their demands. For each $l \in N$, let $d^l = (d_1^l, d_2^l, \cdots, d_T^l)^T$ be the known demand vector of player l. For any given subset of players $J \subset N$, let d^J be the demand vector of J. We assume that all costs for each manufacturer are fixed in a single period. The manufacturers can produce either individually or jointly by keeping inventory at one warehouse, resulting in the MLELSG, specified by (N, V), where the grand coalition is the set of manufacturers N, and the characteristic cost function $V(J) = C(d^J), J \subset N$. The objective is to design a cost-sharing method that allocates the total cost to the different players, that is, computes the cost share $\eta(J, j)$ for each player $j \in J$.

We are interested in cost sharing methods that satisfy *fairness, group-strategyproof, cross-monotonicity, competitiveness and approximate cost recovery*. Fairness implies that no subset I of coalition is charged more than the cost of subset I, i.e., $\sum_{j \in I} \eta(J, j) \leq$

$V(I), \forall I \subset J$. A cost-sharing method is *group-strategyproof* if players has no incentive to collude. It is said to be *cross-monotonic* if the cost share of each player never goes up as the set of participating players increases, that is $\eta(J,j) \geq \eta(S,j)$ for all $j \in J \subset S$. It is *competitive* if the players are charged no more than the true cost, i.e., $\sum_{j\in J} \eta(J,j) \leq V(J)$. And further, if $\sum_{j\in J} \eta(J,j) \geq V(J)/\gamma$, where $\gamma \geq 1$, the cost sharing method is γ-*approximate cost recovering* . It is *exact cost recovering* if $\gamma = 1$. It follows from Pál and Tardos [10] that competitiveness and cross-monotonicity together imply fairness. Moulin and Shenker [9] show that cross monotonic cost-sharing leads to group strategyproof mechanisms. So it is sufficient to devise an approximate cost recovery cost-sharing method that is cross-monotonic and competitive.

Van den Heuvel et al. [11] analyze a special case of the 1-level ELS game, where backlogging is not allowed and the ordering cost includes a fixed setup cost and a linear cost. They show for this special case that the core is always nonempty. Chen and Zhang [4] focus on the core of the 1-level ELS game under more general conditions by allowing for backlogging and concave ordering cost. They show that an allocation in the core can be computed in polynomial time by solving a linear program. Utilizing the equivalence between the economic lot-sizing problem and the facility location problem (FLP) [12], Xu and Yang [14] propose a cross-monotonic competitive 3β-approximate cost-sharing method for the 1-level ELS game, where $\beta \geq 1$ is a constant. Yang et al. [16] present a cost-sharing method for the soft-capacitated ELS game.

It is well-known that the MLELSP can be equivalently formulated as the k-level FLP [12]. We review briefly the literature about the k-level FLP and its associated games. Aardal et al. [1] show that the k-level FLP can be solved within a factor of 3 by a linear programming (LP) rounding algorithm, and later Zhang [18] improves this result to 1.77 for the 2-level FLP. The best combinatorial algorithm with a performance factor of 3.27 for the k-level FLP is due to Ageev et al. [2]. Xu and Du [13] present a 6-approximate cost-sharing method for the k-level facility location game. With respect to the basic FLP and its variants, we refer to [3, 5, 8, 15, 19] and the references therein.

The main contribution of this paper is to develop a cost-sharing method for the MLELSG that is cross-monotonic, competitive, and $2\beta(2\beta+1)$-approximate cost recovery, where $\beta \geq 1$ is a constant and will be addressed at Assumption 1. The rest of the paper is organized as follows. We present our algorithm and its analysis in Section 2. Some discussions are offered in Section 3.

2 The cost-sharing method

In this section, we present our cross-monotonic cost-sharing method for the MLELSG by adopting the ghost-process, developed first in [10].

For any coalition J, the characteristic cost function $V(J)$ is the optimal value of the following integer program.

$$
\begin{array}{llll}
V(J) := \min & \sum_{t=1}^{T} \sum_{p\in\wp} d_t^J q(tp) x_{tp} + \sum_{i=1}^{k} \sum_{s_i=1}^{T} f_{i,s_i} y_{i,s_i} & & \\
\text{s.t.} & \sum_{p\in\wp} x_{tp} = 1, & \forall t = 1,\cdots,T, & \\
& \sum_{p:(i,s_i)\in p} x_{tp} \le y_{i,s_i}, & \forall t, s_i = 1,\cdots,T; i = 1,\cdots,k & (1)\\
& x_{tp} \in \{0,1\}, & \forall p \in \wp, t = 1,\cdots,T, & \\
& y_{i,s_i} \in \{0,1\}, & \forall s_i = 1,\cdots,T; i = 1,\cdots,k. &
\end{array}
$$

In the above program, $q(tp)$ is the cost of satisfying one unit demand at period t by the production schedule $p = \{(1,s_1),(2,s_2),\cdots,(k,s_k)\}$, namely,

$$q(tp) = c_{t,s_1} + \sum_{i=1}^{k-1} c_{s_i,s_{i+1}} = c_{t,s_1} + c(p),$$

$$
c_{s_{i-1},si} = \begin{cases} p_{i,s_i} + \sum_{s=s_i}^{s_{i-1}-1} h_{i,s}, & \text{if} \quad s_i \le s_{i-1}, \\ p_{i,s_i} + \sum_{s=s_{i-1}}^{s_i-1} g_{i,s}, & \text{otherwise,} \end{cases}
$$

where $i = 1,\cdots,T$ and $s_0 = t$. In addition, the indicator variable $x_{tp} = 1$ if and only if the demand at period t is satisfied by p, and $y_{i,s_i} = 1$ if and only if a batch of productions is processed at period s_i in level i. The first constraint of problem (1) indicates that the demand at each period must be met by some production schedule p, and the second one shows that if the demand at period t is supplied by production period s_i in level i, then a production batch is processed at this production period.

The dual of the linear program relaxation of (1) is

$$
\begin{array}{lll}
\max & \sum_{t=1}^{T} \alpha_t d_t^J & \\
\text{s.t.} & \alpha_t - \sum_{s_i\in p} \theta_{s_i,t} \le q(tp), & \forall p \in \wp, t = 1,\cdots,T, \\
& \sum_{t=1}^{T} d_t^J \theta_{s_i,t} \le f_{i,s_i}, & \forall s_i = 1,\cdots,T; i = 1,\cdots,k, \\
& \theta_{s_i,t} \ge 0, & \forall t, s_i = 1,\cdots,T; i = 1,\cdots,k.
\end{array}
$$

Let $\bar{T} = \{1,\cdots,T\}$. For any $s_i, t \in \bar{T}$, we always use s_i as a possible production period in level i and t as a time period. Finally we denote by p_{s_k} a production path with the ending production period s_k of level k. In order to present our algorithm, we define the following notations:

$$q(s_k,s_k') = \min_{t\in\bar{T}, p_{s_k}, p_{s_k'}\in\wp} q(tp_{s_k}) + q(tp_{s_k'}),\ q(s_k,t) = \min_{p_{s_k}\in\wp} q(tp_{s_k}).$$

In general, the quantities defined in the above may not satisfy the following triangle inequalities

$$q(s_k,t) \le q(s_k',t) + q(s_k,s_k'), \text{ and } c_{t,s_1} \le q(tp_{s_i}) + q(tp_{s_i}')$$

where two paths p_{s_i}, p'_{s_i} are respectively defined by

$$p_{s_i} = \{(1,s_1),(2,s_2),\cdots,(i,s_i)\}, \quad \text{and} \quad p'_{s_i} = \left\{(1,s'_1),\cdots,(i-1,s'_{i-1}),(i,s_i)\right\}. \quad (2)$$

Since the triangle inequalities are important for the analysis of our algorithm, we impose the following mild condition for the rest of the paper.

Assumption 1. For any $t,t' \in \bar{T}$, $i,l = 0,1,\cdots,k$, we assume that $h_{l,t} \le \beta h_{i,t}$, $g_{l,t} \le \beta g_{i,t}$, $\frac{1}{\beta} g_{i,t} \le h_{i,t} \le \beta g_{i,t}$, $p_{i,t'} \le \beta p_{i,t}$, where $\beta \ge 1$ is a constant.

Assumption 1 implies the following simple observation.

Lemma 1. *For any possible production periods s_k, s'_k in level k, any time periods t,t', we have*

$$q(s_k,t) \le \beta\left(q(s'_k,t) + q(s_k,t') + q(s'_k,t')\right),$$
$$c_{t,s_1} \le \beta\left(q(tp_{s_i}) + q(tp'_{s_i})\right),$$

where two paths p_{s_i}, p'_{s_i} are the same as (2).

From the arbitrariness of t' in Lemma 1 and the definition of $q(s_k,s'_k)$, we obtain the *weak triangle inequality*

$$q(s_k,t) \le \beta\left(q(s'_k,t) + q(s_k,s'_k)\right).$$

Now we present our algorithm as follows.

Algorithm 1

Step 1 (The ghost process) We introduce the notion of *time* $\tilde{t}$ advancing from 0 to $+\infty$. The *ghost* of demand period t is a ball centered at t with radius $\tilde{t}$ at *time* $\tilde{t}$. The contribution of the ghost of demand period t towards filling the production period (i,s_i) is $\theta_{s_i,t}$ which will be updated over time $\tilde{t}$. Set $\theta_{s_i,t} = 0$ at time $\tilde{t} = 0$.

- A production period (i,s_i) is said to be *fully paid* at some time $\tilde{t}(i,s_i)$ if $\sum_{t=1}^{T} d_t^J \theta_{s_i,t} = f_{i,s_i}$. Now it is time to leave the production period (i,s_i) for each demand period t with $\theta_{s_i,t} > 0$.
- A path is *fully paid* if and only if every production period on the path is fully paid.
- The ghost of demand period t is said to *touch* production period $(i,s_i)(1 \le i \le k)$ at time $\tilde{t}$, if $q(tp) + \sum_{l=1}^{i-1} \theta_{s_l,t} = \tilde{t}$ for some fully paid path $p = \{(1,s_1),\cdots,(i,s_i)\}$. When the ghost of demand period t touches a fully paid production period (i,s_i) at time $\tilde{t}$, the time $\tilde{t}$ is also called the moment that the ghost t *leaves* the production period (i,s_i). And further, if the ghost of demand period t touches a production period (i,s_i) that is not fully paid at time $\tilde{t}$, we *start increasing* $\theta_{s_i,t}$ with unit speed.

Step 2 (The cost shares) Let $R_{i,s_i} = \{t \in \bar{T} | \theta_{s_i,t} > 0\}$ be the set of demand periods that contribute towards filling the production period (i,s_i). The successor of (i,s_i) is the production period in level $i-1$ through which (i,s_i) is touched by a ghost for the first time, i.e., $\text{succe}(i,s_i) := \arg\min_{s_{i-1}\in\bar{T}} \tilde{t}(i-1,s_{i-1}) + c(s_i,s_{i-1})$. The successor of $(1,s_1)$ is its closest demand period. For any production period (k,s_k), we say that $p(s_k) = \{(1,s_1),\cdots,(k,s_k)\}$ is the associated path of (k,s_k) if $(l,s_l) = \text{succe}(l+1,s_{l+1})(l = 1,\cdots,k-1)$. The *neighborhood* of (k,s_k) is the set of demand periods contributing to the associated path $p(s_k)$, denoted as
$N(k,s_k) = \{t \in \bar{T} | \theta_{s_i,t} > 0, \text{ for some } (i,s_i) \in p(s_k)\}$. Let

$$\alpha_t = \min\left\{ \min_{t\notin N(k,s_k)} \min_{p_{s_k}\in\wp} \left(q(tp_{s_k}) + \sum_{(i,s_i)\in p_{s_k}} \theta_{s_i,t} \right), \min_{t\in N(k,s_k)} \tilde{t}(k,s_k) \right\}.$$

The cost share of each player $j \in J$ is $\eta(J,j) = \sum_{t=1}^{T} d_t^j \alpha_t$.

Step 3 (The production periods) Sort all the possible production periods of level k in a nondecreasing order of the fully paid time $\tilde{t}(k,s_k)$. According to this order, we open the production period (k,s_k) and the associated path $p(s_k)$ if and only if there is no already open production period (k,s'_k) such that $q(s_k,s'_k) \le 3\tilde{t}(k,s_k)$.

Step 4 (The demands assignment) Suppose that (k,s_k) is open, we assign the demand in $N(k,s_k)$ to the associated path. The remaining demands will be assigned to the closest open path.

One can show that Algorithm 1 is a well-defined polynomial time combinatorial algorithm. The cross-monotonic cost share is given at Steps 1 and 2 of Algorithm 1. In order to bound the approximate factor of cost recovery, we construct a primal integer feasible solution of problem (1) at Steps 3 and 4 of Algorithm 1.

Theorem 2. *The cost share generated by Algorithm 1 is a $2\beta(2\beta+1)$-approximate, cross-monotonic, and competitive cost-sharing method for the MLELSG.*

A thorough proof of Theorem 2 will appear in the full version of the paper.

3 Discussions

In this paper, we design a cost-sharing strategy for the MLELSG that is cross-monotonic, competitive, and $2\beta(2\beta+1)$-approximate cost recovery. This result extends some existing ones in the literature. When $\beta = 1$, MLELSG is equivalent to the metric k-FLG, and the 6-approximate cost recovery method for k-FLG in Xu and Du [13] is therefore a special case of our approximate factor. Moreover, since MLELSG contains 1-ELSG as a special case, Algorithm 1 also extends the work of Xu and Yang [14]. As for future research, one potential is to consider the ELSG with general concave ordering cost.

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