The Tenth International Symposium on Operations Research and Its Applications (ISORA 2011)
Dunhuang, China, August 28–31, 2011

Subordinated Gaussian Processes, The Log-Return Principles

Stergios B. Fotopoulos[1] Yu-Xing Luo[1]

[1]Washington State University

Abstract The aim of this article is to provide new results concerning the distribution of the first time the process reaches its ultimum maximum. This is established for the case when the underlying process is either a linear Brownian motion with negative drift or a subordinated Gaussian process. To attain this, one revisits the Wiener-Hopf factorization identities provided by Pecherskii-Rogozin (1969). The Pecherskii- Rogozin identities bring more insight of how quantities like the ascending and descending epochs and the first time to its ultimum supremum inter-play to derive the explicit distribution of the argmax of the process. Extensive simulations are performed to show the closeness between the theoretical and its empirical distribution. We apply our methodology to three USA indices and we show its applicability. We finally provide more evidence that the asset price movement model follows a geometric subordinated Gaussian process.

Keywords Wiener-Hopf factorization; Supremum functional; Ascending and descending epochs; Spitzer's condition; Linear Brownian motion; Generalized hyperbolic distribution; EM algorithm

1 Introduction

The assumption that time series of financial data follows a geometric Brownian motion (Merton 1973, Black and Scholes 1973) became quite popular three to four decades ago. This was due to the fact that normal distribution or Gaussian processes, in general, have been documented in an enormous number of articles and the distribution and the process itself generate nice analytic properties which are applicable to a huge number of applications. Further, the statistical properties of the fluctuations of financial prices have been widely researched since Mandelbrot (1963) and Fama (1965). However, it has been shown that contrary to Gaussian processes, logarithmic prices of assets, i.e., $lnP_t, t \geq 0$, are neither Gaussian nor homogeneous and do not have independent increments (Cont 2001). Despite empirical evidence, classical diffusion models continue to be used as a benchmark due to their simplicity. Preserving the simplicity feature, researchers develop financial models maintaining both time and space homogeneity, i.e., they assume that the log-prices, lnP_t , follow a Lévy process. This class of processes is much broader than the Gaussian counterpart and includes both Gaussian and non-Gaussian processes.

Returning to the statistical properties, Mandelbrot (1963) and Fama (1965) suggested that logarithmic prices follow a stable law with an index close to 1.7. De Vries (1994), Muller et al (1999) and Plerou et al (1999), among others, have shown that there is enough evidence that a Pareto type of distribution could fit well with log-returns with an index, now, falling between 2 and 4. More recently, Barndorff-Nielsen (1997), and Eberlein et al (1998) have provided another class of distribution that fits well with the empirical distribution of log-returns, the generalized hyperbolic (GH) family of distributions. It is well known that the normal inverse Gaussian (NIG) is a member of GH. Therefore, opinions vary among scientists with respect to the shape of the distribution of log-returns. Nevertheless, the results of more than half a century of empirical studies on financial prices indicate that asset prices do share some quite non-trivial statistical properties. Such properties, common across a wide range of instruments, markets and time periods are called stylized empirical evidence (see, e.g., Cont 2001). Some of the stylized statistical properties of the log-returns are the following: (a) the log returns $R_{t,h} = lnP_{t+h} - lnP_t$ have insignificant autocorrelations, i.e.,$R_{t,h} \cong R_h$ (stationary), (b) they exhibit space homogeneity (identically distributed for equally spaced increments), (c) they do not follow Gaussian distribution (specifically, log-returns exhibit kurtosis and fat tail), (d) they are asymmetric, and (e) the logreturns exhibit stochastic volatility with suppressed and cluster periods. Calling upon the above properties, the aim of this article is to develop a methodology to identify a general model such that the underlying process captures all the above robust properties that log-returns share. The proposed model that log-prices (weekly, daily), $lnP_t, t \geq 0$, is then assumed to satisfy both time and space homogeneity. As a note, it has been observed that the normality assumption can be quite reasonable for aggregated data (monthly). Then, the proposed generalized model should be flexible enough to accommodate even this special case.

Another aim of the proposed study is to capture an important property that alarms the majority of investors in a recession period. Specifically, we are interested in studying the behavior of the waiting time that the log-price process takes to reach its maximum. In contrast to the log-prices considered earlier, the observations here are required to satisfy a certain market characteristic (recession period), i.e., it is mandatory that the observations are collected at a period when this characteristic is satisfied. For this specific scenario, we propose to obtain the distribution of the time it takes for the process to reach its maximum. This, in turn, will play a role in the decision of an investor whether or not to stay or remove herself/himself from the market.

The above applications will be shown that they have a strong relationship with fluctuation theory. The classical fluctuation theory has proved vastly fruitful in both theory and applicability. Over the last decade or two, much effort has been devoted to the development of a corresponding theory in continuous processes and particularly for Lévy processes.

In this article we shall make methodical use of the Wiener-Hopf factorization for Lévy processes from which many identities are deduced that play a significant role in the development of fluctuation theory. Among many available tools, the Pecherskii and Rogozin (1969) identity, blending with various techniques used for the development of ascending and descending ladder epochs and heights in continuous-time, will play an important role in obtaining a much greater degree of understanding and expansion of existing theory. It should be highlighted that a continuous version of fluctuation theory is not quite a straight forward imitation of the discrete case, random walks. Clearly, the basic technique applied in random walk theory (combinatorics arguments) is not practically suitable for the continuous case. This work is formulated based largely on the Pecherskii and Rogozin (1969) identity. The key proposal here is to study excursions away from the maximum for Lévy processes. These types of properties coincide with studying excursions away from reflected processes. The main objective of this article is then to show the relevance to various applications including financial models.

The outline of the article is as follows: Section 2 presents all constructions at a more general level along with conditions under which they may be implemented. Section 3 considers the case of a linear Brownian motion and obtains explicit expressions for the general cases considered earlier. Section 4 provides explicit results of subordinated Gaussian processes that have close ties with financial models. Finally, Section 5 demonstrates the validity of the results by providing extensive simulations for certain situations of interest and we provide sufficient evidence that three USA indices satisfy the proposed methodologies. Specifically, we analyze time series data for S&P 500, NASDAQ and NYSE from the last three years.

2 Fluctuation of Lévy processes

An important tool in the study of fluctuations is the Wiener- Hopf factorization which we assess below in order to have better insight for some quantities of interest. For more detailed results on the subject, we refer the reader to Sato (1999, Ch 9) and Bertoin (1996).

Let $\{X_t : t \geq 0\}$ be an R -valued Lévy process defined on a filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, P)$, where the filtration $\{\mathscr{F}_t\}$ is assumed to satisfy the usual conditions of right continuity and completion. We take $P(X_0 = 0) = 1$. It is well known from the Lévy-Khintchine formula that a Lévy process on $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, P)$ satisfies $E[exp(i\theta X_t)] = e^{t\psi(\theta)}, t \geq 0, \theta \in R$, where

$$\psi(\theta) = i\mu\theta - \frac{1}{2}\sigma^2\theta^2 + \int_{-\infty}^{\infty}\{1 - e^{i\theta x} + ix\theta I(|x| < 1)\}\Pi(dx) \tag{2.1}$$

with$\mu \in R, \sigma \geq 0$ and Π on R $\setminus\{0\}$ being the Lévy measure. It is also required that the Lévy measure satisfies $\int_{-\infty}^{\infty}\{x^2 \wedge 1\}\Pi(dx) < \infty$. The function ψ is referred to as a Lévy exponent, defined for $\mathscr{R}(\theta) \geq 0$, and its restriction to the non negative real line is strictly convex and satisfies $lim_{\theta\to\infty}\psi(\theta) = \infty$. It is conventional that any Lévy process can be characterized by the triplet (μ, σ, Π).

Let $\overline{X}_t := sup_{u\leq t}X_u$ and $\underline{X}_t := inf_{u\leq t}X_\mu$ denote the supremum and the infimum of the processes X_t , $t \geq 0$, respectively. One can then define the reflected process by $R_t := \overline{X}_t - X_t$. The symbol τ represents a random variable independent of the process $\{X_t : t \geq 0\}$ and is distributed according to exponential distribution with parameter $q > 0$. We further define the first passage times to either maximum or minimum by

$$T_x^+ : inf\{t > 0 : X_t > x\} \qquad \text{and} T_x^- : inf\{t > 0 : X_t \leq x\}, \text{for} \qquad x \in R$$

When $x = 0$,$T_0^+$and T_0^- present the first ascending and first descending ladder epochs, respectively.

The next set of definitions and results are introduced to show relationships between the index time $J(t) = inf\{u \in [0,t] : X_u = \overline{X}_t\}$ that was introduced by Bingham (1975) and the indices introduced below by Pecherskii and Rogozin (1969). Specifically, Pecherskii and Rogozin (1969) found it convenient to study Levy processes using $X_t^{+*} : max\{X_{t-}, X_t\}(X_t^{-*} : min\{X_{t-}, X_t\})$, instead of $\{X_t : t \geq 0\}$. Next, they introduced the modified first passage time to be $T_x^{*+} := inf\{t > 0 : X_t^{*+} \geq x\}(T_x^{-*} := inf\{t > 0 : -X_t^{-*} \geq x\})$. Finally, we define the times it takes the process $X_t^{++} X_t^{-*}$, to attain its maximum for the first and last time by $J^+(t) := inf\{u \in [0,t] : X_u^{+*} = \overline{X_t}\}$ and $J^{+*}(t) := sup\{u \in [0,t] : X_u^{+*} = \overline{X_t}\}$, respectively. Using duality arguments, one may also define in the same way the indices $J^-(t)$ and $J^{-*}(t)$, for the case when the minimum is attained. Calling upon the fact that a LÍęvy process is time and space homogeneous and given that conditions A-C are satisfied, as shown below (except for type A_0), Pecherskii and Rogozin (1969) have then shown that (i.) $P(T_x^{*+} - T_x^+ > 0$ for any $x > 0$ and (ii.) $J^+(t) = J^{*+}(t)$a.s. for all $t \geq 0$. Using the duality property, it can be also shown that (i.) $P(T_x^{-*} - T_x^- > 0$ for any $x > 0$ and (ii.) $J^-(t) = J^{-*}(t)$a.s. for all $t \geq 0$. Thus, Pecherskii and Rogozin (1969) have shown that the time taken to attain the maximum for the first or last time is almost surely the same.

The following theorem reviews the Wiener-Hopf factorizations found in Bingham (1975) and Doney (2005). Similar versions of these factorizations were first introduced in Rogozin (1966) and Pecherskii and Rogozin (1969) under the restricted processes. Comments about the equivalence of the results expressed in the following theorem, the definitions stated above and the original results obtained by Pecherskii and Rogozin (1969) are clarified below.

Theorem 1. *Let $\{X_t : t \geq 0\}$ be a real-valued Lévy process and let τ denote a random variable independent of the process distributed according to exponential distribution with parameter $q > 0$. Then, for $\theta, \vartheta > 0$, we have the following identities and factorizations:*

1. $E[exp(-\vartheta\tau + \theta X_\tau)] = \frac{q}{q+(\vartheta-\psi(\theta))}$
2. $\frac{q}{q+(\vartheta-\psi(\theta))} = \psi_q^+(\vartheta,\theta)\psi_q^-(\vartheta,\theta)$

   where i. $\psi_q^+(\vartheta,\theta)$ is analytic in the half-plane $\Re(\theta) < 0$, continuous and non-vanishing in $\Re(\theta) \leq 0$, and is the Laplace-transform of an infinitely divisible probability on the right half-line, ii. $\psi_q^-(\vartheta,\theta)$ is analytic in the half-plane $\Re(\theta) > 0$, continuous and non-vanishing in $\Re(\theta) \geq 0$, and is the Laplace-transform of an infinitely divisible probability on the left half-line, and iii. subject to i. and ii., the factorization is unique to within factors of the form e^{cs}.
3. *For $q > 0$, $\vartheta > 0$ and $\Re(\theta) \leq 0$*

   $\psi_q^+(\vartheta,\theta) = E[exp(-\vartheta G_\tau^+ + \theta\overline{X}_\tau)] = exp\{\int_0^\infty\int_0^\infty e^{-qt}(e^{-\vartheta t+\theta t} - 1)P(X_t \in dx)\frac{dt}{t}\}$

   where $G_t^+ := sups \leq t : X_s = \overline{X}_s$.
4. *For $q > 0$, $\vartheta > 0$ and $\Re(\theta) \leq 0$*

   $\psi_q^-(\vartheta,\theta) = E[exp(-\vartheta(\tau - G_\tau^+) + \theta(\overline{X}_\tau - X_t))] = exp\{\int_{-\infty}^0\int_0^\infty e^{-qt}(e^{-\vartheta t+\theta t} - 1)$
   $P(X_t \in dx)\frac{dt}{t}\}$
5. *For $q > 0$, $\vartheta > 0$, $\Re(\theta) \leq 0$ and $\Re(\lambda) \geq 0$*

   $\psi_q^+(0,\theta)\psi_q^-(0,\lambda) = E[exp(\theta\overline{X}_\tau - \lambda(\overline{X} - X_\tau))]$.
6. *For $q > 0$, $\vartheta > 0$, and$\Re(\theta) \leq 0$, $\Re(\lambda) \geq 0$ and $\Re(\nu) \geq 0$*

   $E[exp(\theta\overline{X}_\tau - \lambda(\overline{X} - X_\tau) - \nu J(\tau))] = \psi_{q+\nu}^+(0,\theta)\psi_q^-(0,\lambda)exp(\int_0^\infty \frac{e^{-qt}dt}{t}(e^{-\nu t} - 1)$
   $P(X_t > 0))$

The aim of the article is restricted to understanding the behavior of the time $J^+(t)$, the index at which the process reaches its maximum for the first time by time t. Note that when $J^+(t)$ is properly standardized, $J^+(t)$ converges to the arc-sine law. We investigate $J^+(t)$ when t is infinite without any standardization and when the process is drifting to $-\infty$. It is noted that the distribution of the maximum cannot be expressed by elementary functions, except in some particular cases. However, the asymptotic behavior of its maximum, $T_x^+, R_t := \overline{X}_t - X_t$, $J(t)$ and so on, are well established in the literature. Most of these results are done under the random walk scenario, even though the last decade or two there has been increased use of the Lévy processes. Our goal here is to offer explicit expressions of some special cases of their distributions and study their behavior compared to the asymptotic results available. These kinds of properties are unnoticeable in the literature.

Pecherskii and Rogozin (1969) and later Sato (1999) classified Lévy processes $X_t : t \geq 0$ according to the triplet (μ, δ, Π). They identified the following types of processes:

1. A Lévy process is of type A if $\sigma = 0$ and $\nu(R) < \infty$ and $P(X_1 > 0) > 0$, $P(X_1 < 0) < 0$
2. A Lévy process is of type B if $\sigma = 0$, $\nu(R) < \infty$ and $\int_D |x \mid \nu(dx) < \infty$ and
3. A Lévy process is of type C if $\sigma \neq 0$ or $\int_D |x \mid \nu(dx) = \infty$, where $D = \{x \in R : |x| \leq 1\}$.

When the process $\{X_t : t \geq 0\}$ is of type A satisfying $\psi(\theta) = \gamma_0\theta + \int_\infty^\infty \{1 - e^{i\theta x}\Pi(dx)\}$ with $\gamma_0 = 0$, Pecherskii and Rogozin (1969) introduced the subtype A_0. The parameter above represents the drift of the process.

The terminology below originated from Chung and Doob (1965), which will be our basic reference in what follows. To introduce the following proposition, we now refer the reader to Karatzas and Shreve (1988, p.5).

PROPOSITION 1. If a stochastic process $X_t : t \geq 0$ is right-continuous or left-continuous and is adapted to the filtration $\{\mathscr{F}_t\}$, then it is also progressively measurable with respect to $\{\mathscr{F}_t\}$.

To make sense of the implication of the above proposition, we let $n \in \mathbf{N} \backslash \{0\}$,and $k = 0, 1, ..., 2^n - 1$ and we define

$$X_k^n(t) := X_{k/2^n}, fort \in [\frac{k}{2^n}, \frac{k+1}{2^n}),$$

and $X_k^n(0) = X_0 = 0$. The constructed map $(k, \omega) \mapsto X_k^n(\omega)$, from $[0, \infty) \times \Omega$ into R is then evidently $\mathscr{B}([0,\infty)) \bigotimes \mathscr{F}_t-$ measurable. By right-continuity, one obtains that $lim_{n\to\infty} X_k^n(t) = X_t$ for any $t \in [\frac{k}{2^n}, \frac{k+1}{2^n})$. Thus, once again, the map from $t \mapsto X_t$ is clearly $\mathscr{B}([0,\infty)) \otimes \mathscr{F}_t-$ measurable.

The main premise below is to show that random walks and Lévy processes are often conflated. The proposed approach utilized here is similar to that of Pecherskii and Rogozin (1969). Here, we just restrict our study to the process $J(t), t \geq 0$, and we obtain an identity using similar tools as the above authors. It is understood that our conclusion will be a special case of that obtained under the general Theorem 1. The contribution here lies in narrowing the work precisely to the index $J(t), t \geq 0$. In what follows and for specific types of Lévy processes, we demonstrate an identity between $J^+(t)$ and G_t^+.

It is known from the discrete case (see,e.g., Feller VII, 1971, p.416) that

$$P(s)=\sum_{n=0}^{\infty}s^n p_n = exp(\sum_{n=1}^{\infty}\frac{s^n}{n}P(S_n>0)) \quad \text{and}$$
$$Q(s)=\sum_{n=0}^{\infty}s^n q_n = exp(\sum_{n=1}^{\infty}\frac{s^n}{n}P(S_n\le 0)), s\in(0,1), \tag{2.2}$$

where $P(s)$ and $Q(s)$ represent probability generating functions of the corresponding sequences $p_n = P(\bigcap_{j=1}^{n}(S_j>0))$, $q_n = P(\bigcap_{j=1}^{n}(S_j\le 0))$, respectively and S_j, $j\ge 1$ denotes partial sums of i.i.d. random variables. Let J_n^+, $n\ge 1$ denote the discrete analogue of $J^+(t), t\ge 0$. It can be seen that

$$P(J_n^+ = j) = p_j q_{n-j}, j=0,1,...,n,n\ge 1, \tag{2.3}$$

where $p_n = P(T_0^- > n)$ and $q_n = P(T_0^+ > n)$, $n\ge 1$. It is then clear that for $s,x\in(0,1)$,

$$\begin{aligned}(1-x)\sum_{n=0}^{\infty}x^n E[s^{J_n^+}] &= (1-x)\sum_{n=0}^{\infty}x^n\sum_{j=0}^{n}s^j p_j q_{n-j}\\ &= (1-x)\sum_{j=0}^{\infty}(sx)^j p_j \sum_{n=j}^{\infty}x^{n-j}q_{n-j} = (1-x)P(sx)Q(x)\\ &= exp(-\sum_{n=1}^{\infty}\frac{x^n}{n}(1-s^n)P(S_n)>0) \end{aligned} \tag{2.4}$$

As in Pecherskii and Rogozin (1969), we set $x=e^{-q/2^n}, s=e^{-v/2^n}$ and $S_k\equiv X_k^n(t)$ for $t\in\left[\frac{k}{2^n},\frac{k+1}{2^n}\right)$ and $n\in N\backslash\{0\}$, and $k=0,1,\dots,2^n-1$. Upon Proposition 1, we let $n\uparrow\infty$. Thus, the left-hand-side of (2.4) tends to the following limit:

$$\begin{aligned}(1-x)\textstyle\sum_{n=0}^{\infty}x^n E[s^{J_n^+}] &= \frac{\left(1-e^{-q/2^n}\right)}{(1/2^n)}\textstyle\sum_{k=0}^{\infty}e^{-q^k/2^n}\frac{1}{2^n}E[e^{-vJ_k^+/2^n}]\\ &\to q\int_0^{\infty}e^{-qu}du E[e^{vJ^+(u)}]. \end{aligned} \tag{2.5}$$

Similarly, letting $n\uparrow\infty$, the right-hand-side of (2.4) converges to the following:

$$\begin{aligned}exp\Big(-\sum_{n=1}^{\infty}\frac{x^n}{n}\big(1-s^n\big)\Big)P(S_n>0) &= exp\Big(-\sum_{k=1}^{\infty}\frac{e^{-qk/2^n}}{(k/2^n)}\frac{1}{2^n}\Big(1-e^{-vk/2^n}\Big)P(X_k^n(u)>0)\Big)\\ &\to exp\Big(-\int_0^{\infty}\frac{e^{-qu}du}{u}\Big(1-e^{-vu}\Big)P(X_u>0)\Big). \end{aligned} \tag{2.6}$$

Combining (2.5) and (2.6), the above results reveal the following:

Theorem 2. *Let $\{X_t : t \geq 0\}$ be a real-valued Lévy process and let τ denote a random variable independent of the process distributed according to exponential distribution with parameter $q > 0$. Then, for $\Re(v) \geq 0$, under all types of Lévy processes, except for type A_0,*

$$E\Big[exp(-vJ^+(\tau))\Big] = exp\bigg\{-\int_0^\infty \frac{e^{-qu}du}{u}(1-e^{-vu})P(X_u > 0)\bigg\}.$$

Upon letting $q \to 0$, we also deduce the following.

Theorem 3. *Under the same conditions as in Theorem 2 the process $\{X_t : t \geq 0\}$, for $\Re(v) \geq 0$, satisfies:*

$$E\Big[exp\Big(-vJ^+(\infty)\Big)\Big] = exp\bigg(-\int_0^\infty \frac{du}{u}\Big(1-e^{-vu}\Big)P(X_u > 0)\bigg).$$

This is obviously clear since $q\int_0^\infty e^{-qu}duE\big[e^{vJ^+(u)}\big] = \int_0^\infty e^{-u}duE\big[e^{vJ^+(u/q)}\big] \to E\big[e^{vJ^+(\infty)}\big]$ as $q \to 0$. From Theorem 1(3), letting $\theta \to 0$, we also have that for $\Re(\vartheta) \geq 0$,

$$\psi_q^+(\vartheta,0) = E\big[exp\big(-\vartheta G_\tau^+\big)\big] = exp\bigg\{\int_0^\infty \frac{e^{-qt}dt}{t}\Big(e^{-\vartheta t}-1\Big)P(X_t > 0)\bigg\}$$

which coincides with the result obtained in Theorem 2. This confirms that $J^+(t) = G_t^+$ a.s. for all $t \geq 0$.

In a seminal paper, Greenwood and Pitman (1980) showed that for a random variable $\tau \sim Exp(q)$, which is independent of the process $\{X_t : t \geq 0\}$, the bivariate process (τ, X_τ) can be decomposed to the supremum and its reflection as follows:

$$(\tau, X_\tau) = (G_\tau^+, \overline{X}_\tau) + (\tau - G_\tau^+, X_\tau - \overline{X}_\tau).$$

Using the time reversal, they have then shown that

$$(\tau - G_\tau^+, X_\tau - \overline{X}_\tau) =_D (G_\tau^-, \underline{X}_\tau). \tag{2.7}$$

Equation (2.7) is then utilized to obtain similar results to Theorem 2 or 3 for the reversal process. Specifically, we have:

COROLLARY 1. Let $\{X_t : t \geq 0\}$ be a real-valued Lévy process and let τ denote a random variable independent of the process distributed according to exponential distribution with parameter $q > 0$. Then, for $\Re(v) \geq 0$, under all types of Lévy processes except for type A_0,

$$E\big[exp\big(-vJ^-(\tau)\big)\big] = exp\bigg(-\int_)^\infty \frac{e^{-qu}du}{u}\big(1-e^{-vu}\big)P(X_u < 0)\bigg).$$

Upon letting $q \to 0$, one may also obtain the next result. COROLLARY 2. Under the same condition as in Theorem 2 the process $\{X_t : t \geq 0\}$ and for $\Re(v) \geq 0$, satisfies:

$$E\Big[exp\Big(-vJ^-(\infty)\Big)\Big] = exp\bigg(-\int_0^\infty \frac{du}{u}\Big(1-e^{-vu}\Big)P(X_u < 0)\bigg).$$

The above identities have many important consequences, the first of which follows (see also e.g., Bertoin 1996, Kyprianou, 2004):

Theorem 4. *Let $\{X_t : t \geq 0\}$ be a real-valued Lévy process. Then, (we exclude the case that the Lévy process is of type A_0)*

1. *If $\int_1^\infty \frac{du}{u} P(X_u > 0) < \infty$, then $\overline{\lim}_{t\to\infty} X_t < \infty$ a.s. and $\underline{\lim}_{t\to\infty} X_t = -\infty$ a.s.*
2. *If $\int_1^\infty \frac{du}{u} P(X_u < 0) < \infty$, then $\overline{\lim}_{t\to\infty} X_t = \infty$ a.s. and $\underline{\lim}_{t\to\infty} X_t > -\infty$ a.s.*
3. *If $\int_1^\infty \frac{du}{u} P(X_u > 0) < \infty$, then $P\Big(J^+(\infty) < \infty\Big) = 1$ and $P\Big(J^-(\infty) = \infty\Big) = 1$.*
4. *If $\int_1^\infty \frac{du}{u} P(X_u < 0) < \infty$, then $P\Big(J^+(\infty) = \infty\Big) = 1$ and $P\Big(J^-(\infty) < \infty\Big) = 1$.*

To complete some of the arguments expressed above, we also found convenient to introduce various results taken from Rogozin (1966). The importance of condition $\int_1^\infty \frac{du}{u}$ $P(X_u > 0) < \infty$ was also utilized in Rogozin (1966). Specifically, Rogozin (1966) states the following.

Theorem 5. *(Rogozin 1966)*

1. *With probability one, $sup_{t\in[0,\infty)} X_t = \overline{X} < \infty$ if and only if $\int_1^\infty \frac{du}{u} P(X_u > 0) < \infty$. In this case,*

$$E\Big[exp(\theta\overline{X})\Big] = exp\Big\{\int_0^\infty \int_0^\infty \big(e^{\theta x} - 1\big) P(X_t \in dx) \frac{dt}{t}\Big\}.$$

2. *With probability one, $sup_{t\in[0,\infty)} X_t = \infty$ if and only if $\int_1^\infty \frac{du}{u} P(X_u > 0) = \infty$.*

To match random walks with that of Lévy processes, it is of importance to also see how the Spitzer's condition plays under the continuous scenario. The Spitzer's condition relies on whether or not $t^{-1}\int_0^t \frac{du}{u} P(X_u > 0)$ is finite. Once again, Rogozin (1966) shows the following.

Theorem 6. *(Rogozin 1966)*

1. *If $\int_0^1 \frac{du}{u} P(X_u > 0) = \infty$, then $T_0^+ = X_{T_0^+} = 0$ a.s.*
2. *If $\int_0^1 \frac{du}{u} P(X_u > 0) < \infty$, then $P(T_0^+ = 0) = 0$ and $P(\overline{X} = 0) = exp\Big\{ - \int_0^\infty \frac{dt}{t} P(X_t > 0)\Big\}$.*

Before we confine our work to the case of $J^+(\infty)$, it is worth mentioning the law of $J^+(t)$ under a Lévy process. Clearly, the index $J^+(t)$ is associated with the arc-sine law. It is also accepted that the arc-sine law is contingent upon whether the Spitzer's condition holds. Specifically, it has been shown that when

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t P(X_u > 0)du = \rho \in [0,1], \tag{2.8}$$

then, for $\rho \in (0,1), J^+(t)$ follows the generalized arc-sine law concentrated on $[0,1]$ given by

$$\lim_{t\to\infty} P\bigg(\frac{J^+(t)}{t} \in du\bigg) = \frac{(u)^{\rho-1}(1-u)^{-\rho}}{\Gamma(\rho)\Gamma(1-\rho)} du = \frac{sin\rho\pi}{\pi}(u)^{\rho-1}(1-u)^{-\rho} du, u < 1. \tag{2.9}$$

One may also obtain the same results if the limits in either (2.8) or (2.9) are replaced by $t \downarrow 0$. It should be noted that equation (2.9) holds if one replaces $t^{-1}J^{+}(t)$ either by $t^{-1}G_t^{+}, t>0$ or $\frac{1}{t}\int_0^t P(X_u>0)du$. Moreover, when $\rho = 0$ (or 1), we just have the Dirac point mass at zero (or respectively the Dirac point mass at 1) (see, e.g., Bertoin 1996, p. 169).

The intention of this article is to capture the behavior of the random variable $J^{+}(\infty)$ when $J^{+}(\infty) < \infty$ (finite). This is equivalently translated by saying that the Lévy process is drifting to $-\infty$. To this point, we confine our analysis only to the case when $\rho = 0$ and identifying the law of the index $J^{+}(\infty)$ without any further standardization. While Theorem 3 provides an explicit form of the Laplace transform of $J^{+}(\infty)$, the expression therein is still of technical interest. We still need to have an exact expression of $P(X_u > 0), u > 0$, before we can have an exact expression of the Laplace transform. Upon knowing the distribution of the underlying process we then should be able to obtain the corresponding density of $J^{+}(\infty)$ by inverting the Laplace transform. To this end, we shall concentrate on developing the analysis by assuming that the underlying process is of Gaussian type.

3 An application: linear Brownian motion

Brownian motion is one of the most fundamental distinctive aspects of stochastic processes. It has been documented in a multitude of articles and in a huge number of textbooks. It plays a bench-mark in studying more complex processes. This is exactly the reason we employ it here. We consider the probability law

$$\mu(x;\mu,s) = \frac{1}{\sqrt{2\pi s^2}} exp\left(-(x-a)/2s^2\right),$$

supported on $\mathbf{R}$, where $a \in \mathbf{R}$ and $s > 0$. This is the well-known Gaussian distribution. It is clearly a Levy process (infinitely divisible) with no jumps. It is easy to see that the characteristic exponent of a linear Brownian motion is given by $\psi(\theta) = ia - \frac{1}{2}s^2\theta^2$, i.e., a Brownian motion has the following stochastic representation:$X_t := at + sB_t$, where $\{B_t : t \geq 0\}$ is the standard Brownian motion. It is easy to see that $\{X_t : t \geq 0\}$ is both time and space homogeneous with continuous paths. To illustrate the theory expressed in Section 2, we here assume that $a < 0$.

It is known that $T_x^{+} := \inf\{t > 0 : X_t > x\}$ is the first time a Brownian motion with linear drift $a < 0$ exceeds level $x \in \mathbf{R}$.It is also known that T_x^{+} is a stopping time. Moreover, since the Brownian motion has continuous paths, it is almost certain that $X_{T_x^{+}} = aT_x^{+} + sB_{T_x^{+}} = x$. From the strong Markov property one can then see that $\{X_{T_x^{+}+t} = a(T_x^{+}+t) + sB_{T_x^{+}+t} - x : t \geq 0\} =_{\mathscr{D}} \{X_t : t \geq 0\}$, i.e., the first passage time, T_x^{+} $x \in \mathbf{R}$, can be decomposed to independent components, $T_x^{+} = T_y^{+} + \widetilde{T}_{x-y}^{+}$ with T_y^{+} and $\widetilde{T}_{x-y}^{+}$ being independent and $\widetilde{T}_{x-y}^{+} =_{\mathscr{D}} T_{x-y}^{+}$for $x, y \in \mathbf{R}$. Moreover, it can be shown thatT_x^{+} is a subordinator (infinitely divisible) for $x > 0$, which is characterized by the triplet

$$\mu = -\frac{2s^2x}{a}\int_0^{\mu/s^2} \varphi(y)\mathrm{d}y, \sigma = 0 \text{ and } \prod(\mathrm{d}t) = \frac{x}{\sqrt{2\pi s^2 t^3}} e^{-a^2t/2s^2}\mathrm{d}t,$$

and concentrated on $(0,\infty)$. In addition, the cumulative distribution function of $T_x^{+}$is obtained by

$$P(T_x^+ \le t) = P(\overline{X}_t \ge x) = \Phi\Big(-\frac{x}{s\sqrt{t}} + \frac{a}{s}\sqrt{t}\Big) + e^{2ax/s^2}\Phi\Big(-\frac{x}{s\sqrt{t}} - \frac{a}{s}\sqrt{t}\Big), \tag{3.1}$$

and the law of T_x^+ can be explicitly computed as (see also, e.g., Karatzas & Shreve, 1988, p.197)

$$P(T_x^+ \in \mathrm{d}t) = h(t,x;a,s)\mathrm{d}t = \frac{x\mathrm{d}t}{\sqrt{2\pi s^2 t^3}} e^{xa/s^2} exp\Big(-\frac{1}{2s^2}(x^2t^{-1} + a^2 t)\Big), \tag{3.2}$$

where $\Phi(x) = \int_0^x \varphi(y)\mathrm{d}y = \frac{1}{\sqrt{2\pi}}\int_0^x e^{-y^2/2}\mathrm{d}y$. Note that when $a = 0$, $\rho = 1/2$ for the Brownian motion. Hence the arcsine law is given by

$$P\Big(\frac{J^+(t)}{t} \in \mathrm{d}u\Big) = \frac{u^{-1/2}(t-u)^{-1/2}}{\pi}\mathrm{d}u = \frac{2}{\pi}\arcsin(\sqrt{u})\mathrm{d}u, 0 < u < 1. \tag{3.3}$$

For the discrete case and from equation (2.3), we have that the probability a process reaches its maximum for the first time in the j^{th} step can be expressed using the following identity: $P(J_\infty^+ = j) = P(T_0^- > j)P(T_0^+ = \infty)$. With this in mind, we shall try to derive an explicit expression for the continuous analog when the process follows a Brownian motion with drift. The first result is expressed as follows. From Theorem 6, it is clear that $\int_0^1 \frac{du}{u} P(X_u > 0) = \infty$, and one may easily see that $T_0^+ = X_{T_0^+} = 0$ a.s. and $P(\overline{X} = 0) = 0$.

Theorem 7. *For the Brownian motion $\{X_t = at + sB_t : t \ge 0\}$, for $\Re(v) \ge 0, a < 0$ and $s > 0$, the following holds:*

$$E[exp(-vJ^+(\infty))] = \frac{2}{1 + \Big(1 + \frac{2vs^2}{a^2}\Big)^{1/2}} = \frac{\Big(2\frac{|a|}{\sqrt{2}s}\Big)\Big\{\Big(v + \frac{a^2}{2s^2}\Big)^{1/2} - \frac{|a|}{\sqrt{2}s}\Big\}}{v}.$$

PROOF. From Theorem 3, we begin determining the following:

$$\begin{aligned}
\int_0^\infty \frac{\mathrm{d}u}{u}(1 - e^{-vu})\overline{\Phi}\Big(\frac{-a\sqrt{u}}{s}\Big) &= \int_0^\infty \overline{\Phi}\Big(\frac{-a\sqrt{u}}{s}\Big)\mathrm{d}\Big\{\sum_{j=0}^\infty \frac{(-1)^j(vu)^{j+1}}{(j+1)!(j+1)}\Big\} \\
&= \frac{1}{2\sqrt{\pi}}\Big(\frac{-a}{s\sqrt{2}}\Big)\sum_{j=0}^\infty \frac{(-1)^{j+1}v^{j+1}}{(j+1)!(j+1)}\int_0^\infty u^{j+\frac{1}{2}} e^{-\frac{a^2u}{2s^2}}\mathrm{d}u \\
&= \frac{1}{2\sqrt{\pi}}\sum_{j=0}^\infty (-1)^j \frac{\Gamma(j+\frac{3}{2})}{(j+1)!(j+1)}\Big(\frac{2vs^2}{a^2}\Big)^{j+1} \\
&= \frac{1}{2\sqrt{\pi}}\int_0^{\frac{2vs^2}{a^2}}\sum_{j=0}^\infty (-1)^j \frac{\Gamma(j+\frac{3}{2})j!}{(j+1)!}\frac{u^j}{j!}\mathrm{d}u \\
&= \frac{\Gamma(3/2)}{2\sqrt{\pi}}\int_0^{\frac{2vs^2}{a^2}} {}_2F_1(\frac{3}{2},1;2;-u)\mathrm{d}u \\
&= \frac{1}{4}\int_0^{2vs^2/a^2}\frac{2\mathrm{d}u}{(1+u)^{1/2}\{1+(1+u)^{1/2}\}},
\end{aligned}$$

where ${}_2F_1(1,b;c;z)$ denotes the Gauss hypergeometric function, which converges for arbitrary a,b and c and for real z lying in the unit disc, i.e., $|z|<1$, and for $z=\pm 1$ if $\Re(c-a-b)>0$. Moreover,the last equality above was found in Abramowitz and Stegun (1970, p.556). In completing the algebra above we also noticed that

$$\begin{aligned}\int_0^\infty \frac{\mathrm{d}u}{u}(1-e^{-vu})\overline{\Phi}\Big(\frac{-a\sqrt{u}}{s}\Big) &= \frac{1}{2}\int_0^{2vs^2/a^2}\Big\{\frac{1}{(1+u)^{1/2}}-\frac{1}{1+(1+u)^{1/2}}\Big\}\mathrm{d}u\\ &= \ln\Big\{\Big(1+(1+\frac{2vs^2}{a^2})^{1/2}\Big)/2\Big\}.\end{aligned}$$

In conjunction with Theorem 3, the proof of Theorem 7 is now in order.

To demonstrate the subsequent result, we write $\mathscr{L}(f)(v)=\int_0^\infty e^{-vt}f(t)\mathrm{d}t=\widehat{f}(v), v\in[0,\infty)$,to denote the oneĺCsided Laplace transform on the right-half-line of a function f and we call $\mathscr{L}^{-1}(\widehat{f})(t)=f(t), t\in[0,\infty)$,to denote its corresponding inverse Laplace transform. So, if f is continuous on $t\in[0,\infty)$, and the function $e^{-vt}f(t), t\in[0,\infty)$,is absolutely integrable then $\widehat{f}$ is an analytic function for $\Re(v)\in[0,\infty)$. It is also well known that $\mathscr{L}^{-1}\Big(a\widehat{f}+\frac{b}{v}\Big)(t)=af(t)+b$, for $a,b\in\mathbf{R}$, and $\mathscr{L}^{-1}\Big(\widehat{f}/v\Big)(t)=\int_0^t f(t)\mathrm{d}t$.

In light of the above observations, the following Theorem is in order.

Theorem 8. *For the Brownian motion $\{X_t=at+sB_t : t\geq 0\}$, for $\Re(v)\geq 0, a<0$ and $s>0$, the following holds:*

$$f_{J^+(\infty)}(t)=\frac{1}{\sqrt{\pi}}\frac{a^2}{s^2}\int_{\sqrt{a^2t/2s^2}}^\infty \frac{e^{-y^2}}{y^2}dy.$$

PROOF. In conjunction with the above remarks and the fact that

$$\mathscr{L}^{-1}\Big(\frac{(v+a)^{1/2}}{v}\Big)(t)=\frac{e^{-at}}{\sqrt{\pi t}}+a^{1/2}erf(\sqrt{at}),$$

we obtain that, for $t>0$,

$$\begin{aligned}\mathscr{L}^{-1}\left(2\frac{|a|}{\sqrt{2}s}\frac{\Big(v+\frac{a^2}{2s^2}\Big)^{1/2}}{v}-2\frac{\frac{a^2}{2s^2}}{v}\right)(t) &= 2\frac{|a|}{\sqrt{2}s}\frac{e^{-a^2t/2s^2}}{\sqrt{\pi t}}-\frac{a^2}{s^2}+\frac{a^2}{s^2}erf(\sqrt{a^2t/2s^2})\\ &= \frac{2}{\sqrt{2\pi}}\frac{|a|}{s}\frac{e^{-a^2t/2s^2}}{\sqrt{t}}-\frac{a^2}{s^2}erf(\sqrt{a^2t/2s^2})\\ &= \frac{2}{\sqrt{\pi}}\frac{a^2}{s^2}\left\{\frac{e^{-a^2t/2s^2}}{2\sqrt{a^2t/2s^2}}-\int_{\sqrt{a^2t/2s^2}}^\infty e^{-y^2}dy\right\}\\ &= \frac{1}{\sqrt{\pi}}\frac{a^2}{s^2}\int_{\sqrt{a^2t/2s^2}}^\infty\frac{e^{-y^2}}{y^2}dy.\end{aligned}$$

where $erf(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-y^2}dy\geq 0$. This completes the proof of Theorem 8.

One may also recognize that

$$\int_0^\infty \frac{a^2}{2s^2}\frac{e^{-a^2t/2s^2}}{\sqrt{a^2t/2s^2}}\mathrm{d}t=\int_0^\infty t^{\frac{1}{2}-1}e^{-t}\mathrm{d}t=\Gamma(1/2)=\sqrt{\pi}, \quad \text{and} \tag{3.4}$$

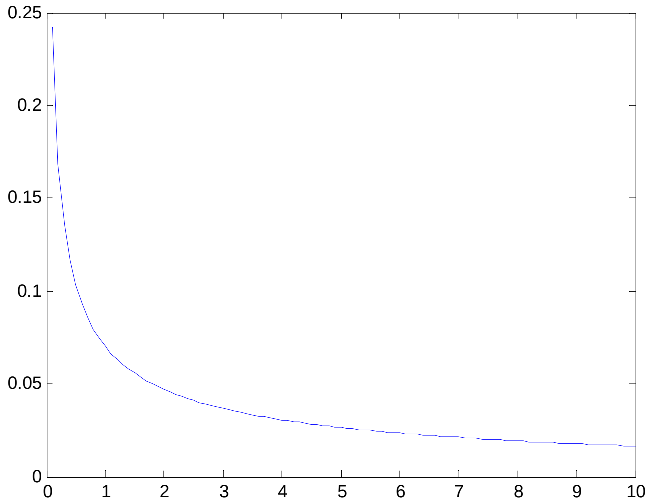


Figure 1: The density plot of $f_{J^+(\infty)}(t)$ under $a = -0.1$ and $s = 1$.

$$\begin{aligned}\frac{a^2}{s^2}\int_0^\infty \mathrm{d}t \int_{\sqrt{a^2t/2s^2}}^\infty e^{-y^2}\mathrm{d}y &= (\frac{a^2}{2s^2})^2\int_0^\infty \mathrm{d}t\int_t^\infty \frac{e^{-\frac{a^2u}{2s^2}}\mathrm{d}u}{\sqrt{a^2u/2s^2}} \\ &= \int_0^\infty u^{\frac{3}{2}-1}e^{-u}\mathrm{d}u = \Gamma(3/2) = \sqrt{\pi}/2. \qquad (3.5)\end{aligned}$$

Combining (3.4) and (3.5), it can then be seen that $\int_0^\infty f_{J^+(\infty)}(t)\mathrm{d}t = 1$. This clearly confirms that $f_{J^+(\infty)}(.)$ is a proper density.

4 Subordinated Gaussian process

In this section we describe how the concepts, methods and results in the previous sections can be applied to provide a rigorous mathematical model for finance. We shall concentrate on the most fundamental issues and those quantities which are most closely related to Sections 2 and 3. The classical model proposed by Samuelson (1965) for a risky asset is generally accepted to be of an exponential linear Brownian motion with drift:

$$P_t = P_0 \exp(at + sB_t), t \geq 0, \qquad (4.1)$$

where $P_0 > 0$ is the initial value of the asset, $\{B_t : t \geq 0\}$ is the standard Brownian motion, $s > 0$ and $a \in \mathbf{R}$. Model (4.1) offers a feature that asset values have multiplicative time and space homogeneity. For example, for any $0 \leq u < t < \infty$, $P_t = P_u \times \tilde{P}_{t-u}$, where $\tilde{P}_{t-u}$ is independent of P_u and $\tilde{P}_{t-u}$ is identically distributed to P_{t-u}. As for the preceding condition being realistic in financial markets, it is clearly open to debate. Nevertheless, for purposes of financial modeling, exponential Brownian motion has proved to be the right model to capture the curiosity of mathematicians, economists and financial practitioners.

To improve statistical accuracy of model (4.1) and thus improve derivative pricing, risk management and portfolio optimization, to name some key areas of application, many extensions of the basic models have been introduced. In this article, we shall emphasize only three points where model (4.1) of risky asset can be shown (empirically) to be inadequate. Specifically, there is concern for log-returns having continuity in their paths and that log-returns of the value of a risky asset are symmetric and follow a normal distribution. It is noticeable that model $\{X_t = at + sB_t : t \geq 0\}$ has continuous paths and therefore cannot accommodate jumps which arguably are present in observed historical data of certain risky assets due to shocks in the market. We again denote the log-returns by $\ln P_{t+h} - \ln P_t = \ln P_{t+h}/P_t = R_{t,h},\ t \geq 0,\ h > 0$. Thus, the most simplistic approach to identify a stochastic behavior of the log-returns is to assume that the increments are independent and normally distributed with mean ha and variance hs^2. However, it has been shown that the standard option pricing model of Black and Scholes (1973) and Merton (1973) is inconsistent with option data for at least the last two decades. Looking at empirical densities of log-returns from financial data, one observes the following stylized features: there is more mass near the origin versus the normal distribution, less at the sides than the normal and considerably more mass in the tails. This means that tiny price movements occur with higher frequency, small and middle sized movements with lower frequency, and big changes (high jumps) are much more frequent than predicted by normal distribution. Another point for consideration which seems to fit well with empirical data is the asymmetric behavior of the log-returns. On the theoretical piece, Carr et al (2001) suggest that price processes for financial assets must have a jump component but they do not need to have a diffusion component. Their argument rests on recognizing that all price processes of interest may be regarded as Brownian motion subordinated to a random clock. These types of models do support the three points mentioned earlier. To this end, recent literature recommends that a possible remedy for these three points is to work with

$$P_t = P_0 \exp(X_t),\ t \geq 0, \tag{4.2}$$

instead of (4.1) where again $P_0 > 0$ is the initial value of the risky asset, and $\{X_t : t \geq 0\}$ is now a Gaussian subordinated process. On the basis of the Geman et al (2001) idea, we propose the following result. We let $\{X_t : t \geq 0\}$ be a linear Brownian motion. Thus, the characteristic function $\hat{v}$ is expressed by

$$k(v) := \ln \hat{\mu}(v) = ias - \frac{1}{2}s^2v^2 \tag{4.3}$$

with $s > 0$ and $a \in \mathbf{R}$. Next, we introduce $\{T_t : t \geq 0\}$ to be a stochastic time change process, which is independent of $\{X_t = at + sB_t : t \geq 0\}$. This, again, is a one-dimensional process. The process $\{T_t : t \geq 0\}$ is called subordinator if it is additive with homogeneous increments, and it is non-decreasing (a.s.). It is known that subordinators are infinitely divisible processes. Moreover, it is recognized that the cumulant generating function of a subordinator satisfies the following representation:

$$E[e^{-uT_1}] := \exp(\psi(-u)) = \exp(ub_0 + \int_0^\infty (e^{-ux} - 1)v_0(\mathrm{d}x)), \tag{4.4}$$

where $b_0 \geq 0$ and v_0 is a Lévy measure. In the present article, we assume that $b_0 = 0$. Then, the subordinator is embedded into the Gaussian process to form the Gaussian subordinated process. It is a Lévy process and will be denoted by $\{X_t^\circ : t \geq 0\}$ (see, e.g., Sato, 1999, Theorem 30.1, p.198). The Lévy and probability measures are expressed as follows:

$$\tilde{v}(B) = \frac{1}{(2\pi s^2)^{1/2}} \int_0^\infty \int_B \exp(-(x-ay)^2/2s^2 y) \mathrm{d}x v_0(\mathrm{d}y),\ B \in \mathscr{F} \text{ and} \tag{4.5}$$

$$\tilde{\mu}(B) = P(X_t^\circ \in B) = \frac{1}{(2\pi s^2)^{1/2}} \int_0^\infty \int_B \exp(-(x-ay)^2/2s^2 y) \mathrm{d}x P(T_t \in \mathrm{d}y),\ B \in \mathscr{F}, \tag{4.6}$$

where B is a Borel set and $\mathscr{F}$ is a σ-algebra. It is easy to show that when $b_0 = 0$ the subordinated Gaussian processes do not have a Gaussian component. To complete the description of subordinated processes, we also add that since $\{X_t^\circ : t \geq 0\}$ is a Lévy process, then its characteristic function satisfies the following representation:

$$\hat{\tilde{\mu}}_t(\theta) = \exp(-t\psi(\theta)),$$

where $\theta \in \mathbf{R}$. Note that since both $\{X_t : t \geq 0\}$ and $\{T_t : t \geq 0\}$ have right continuous paths, then the process $\{X_t^\circ : t \geq 0\}$ also has right continuous paths. Hence, for $0 \leq u < v < w < t < \infty$ and $\theta_1, \theta_2 \in \mathbf{R}$, we have

$$\begin{aligned} &E[\exp(i\theta_1(X_v^\circ - X_u^\circ) + i\theta_2(X_t^\circ - X_w^\circ))] \\ &\quad = E[\exp((i\theta_1 a - \frac{1}{2}s^2\theta_1^2)(T_v - T_u) + (i\theta_2 a - \frac{1}{2}s^2\theta_2^2)(T_t - T_w))] \\ &\quad = E[\exp((i\theta_1 a - \frac{1}{2}s^2\theta_1^2)T_{v-u} + (i\theta_2 a - \frac{1}{2}s^2\theta_2^2)T_{t-w})] \\ &\quad = \exp(-(v-u)\psi(\theta_1) - (t-w)\psi(\theta_2)). \end{aligned} \tag{4.7}$$

Upon (4.7), and the fact that $\{T_t : t \geq 0\}$ has both time and space homogeneity, it implies that $\{X_t^\circ : t \geq 0\}$ has both time and space homogeneity. In light of (4.6), the stochastic process $\{X_t^\circ : t \geq 0\}$ has a stochastic representation given by

$$X_t^\circ = aT_t + sB_{T_t} \equiv X_{T_t},\ t \geq 0. \tag{4.8}$$

To provide some practical insight of the subordinatorąŕs meaning, we offer the following explanation. The link of real time with business time is described through the subordinator T_t. That is to say, one assumes that the value of a given asset follows a process $X^\circ = X \circ T$ because at real time $t > 0$, T_t units of business time have passed and hence the value of the risky asset is positioned at X_{T_t}.

To tie together some of the theory described in Sections 2 or 3, we complete our theoretical framework by presenting new results related to the exact distribution for subordinated Gaussian maxima without any proof. The proofs rely on first conditioning with respect to subordinator and then integrating in terms of the subordinator.

Theorem 9. *Let $\{X_t^\circ : t \geq 0\}$ be a Gaussian subordinated process and the subordinator T_t satisfies, $T_t \to \infty$ a.s. as $t \to \infty$. Then, for $x \geq 0$,*

1. $P(\bar{X}_t^\circ \geq x) = E[\Phi(-\frac{x}{s\sqrt{T_t}} + \frac{a}{s}\sqrt{T_t}) + e^{2ax/s^2}\Phi(-\frac{x}{s\sqrt{T_t}} - \frac{a}{s}\sqrt{T_t})]$,
2. $P(\bar{X}_\infty^\circ \geq x) = e^{2ax/x^2}$,

3. $E[\exp(-vJ^{\circ+}(\infty))] = \frac{(2\frac{|a|}{\sqrt{2}s})\{(v+\frac{a^2}{2s^2})^{1/2}-\frac{|a|}{\sqrt{2}s}\}}{v}$, *and*
4. $f_{J^{\circ+}(\infty)}(t) = \frac{1}{\sqrt{\pi}}\frac{a^2}{s^2}\int_{\sqrt{\frac{a^2 t}{s^2}}}^{\infty}\frac{e^{-y^2}}{y^2}dy, \qquad t \geq 0,$

where $X_t^{\circ}, X_{\infty}^{\circ}$ and $J^{\circ+}(\infty)$ are the analogs of X_t , X , and $J^{+}(\infty)$, respectively, of the previous sections.

It can be noted that the distribution of the ultimum maximum $X_t^{\circ}, t \geq 0$, or the density of $J^{\circ+}(\infty)$ do not depend on the parameters of the subordinator $T_t, t \geq 0$.

5 Simulations, data analysis and concluding remarks

In this section, we wish to assess the performance of the derived asymptotic distribution in Theorem 8 against the empirical distribution under various sample sizes and various parameter choices. For demonstrating the closeness of the two distributions, we find it convenient to utilize the total variation distance

$$d_{TV}(f_\infty, f_n) = sup_{B\in\mathscr{B}}|F_\infty(B) - \hat{F}_n(B)| = \frac{1}{2}\inf_{0}^{\infty}|f_\infty(x) - \hat{f}_n(x)|dx, \qquad (5.1)$$

where $\hat{f}_n(\cdot)$ is the empirical probability distribution function. To exploit the right-hand-side of (5.1), it is practical to employ its discrete version. In particular, we split a large interval,$[0,a_n]$, to equal sized, small in length, as $0 = x_0 < x_1 < \cdots < x_n = a_n < \infty$ such that $x_i = j\frac{h}{a_n}$ with sufficiently small and $a_n \in R$ sufficiently large. Thus, (5.1) can be approximated by

$$d_{TV}(f_\infty, f_n) \approx \frac{1}{2}\{|P(J^{\circ+}(\infty) \leq \frac{h}{2}) - \hat{f}_n(0)|$$
$$+\Sigma_{j=1}^{n}|P(J^{\circ+}(\infty) \in ((j-\frac{1}{2})h, (j+\frac{1}{2})h)) - \hat{f}_n(jh)|\} \qquad (5.2)$$

Simulations are performed by letting the step function and sample sizes be as follows: $h = 0.1$ and n = 1000, 2500 , 5000 and 10000. For each of the above cases, the drift a is chosen to be a = -0.05, - 0.08, - 0.1 and -0.15. For the above cases, we assume that s = 1. Further, for h = 0.1 and n = 500, 1000 , 2000 and 3000, and for the same choices of a , we also consider the case of s = 0.5 . The results, based on 50,000 simulations, are presented in Tables 1 and 2. As one may expect, the results show that there is an excellent agreement between theoretical distribution and the empirical distribution. The total variation distance decreases as the drift decreases.

To illustrate that financial data satisfy the three properties described in Section 4 and the fact that the subordinated Gaussian model indeed fits well with the financial data, we now apply our methodology to three data sets taken from three USA indices. The data sets concerning daily logreturns of S&P 500, NASDAQ and NYSE for the period from January 1, 2008 to December 31, 2010. For all three sets, we fit the NIG distribution. This distribution is a member of a more general family, the GH family. Our preference for the NIG distribution is due to its analytical tractability and the fact that the data appear to fit extremely well. The stochastic representation of each log-return is now represented by the random variable X_1° expressed as $X_1^{\circ} = aT_1 + sB_{T_1}$, where T_1 is inverse Gaussian

	$a=-0.05$	$a=-0.08$	$a=-0.1$	$a=-0.15$
$n=1000$	0.2070	0.1163	0.0840	0.0544
$n=2500$	0.1315	0.0852	0.0745	0.0640
$n=5000$	0.1206	0.0976	0.0881	0.0686
$n=10000$	0.1423	0.1115	0.0942	0.0644

Table 1: Total variation distances of theoretical and empirical distributions (based on 50,000 simulations) of the time the process reaches ultimum maximum when $s=1$

	$a=-0.05$	$a=-0.08$	$a=-0.1$	$a=-0.15$
$n=500$	0.1368	0.0661	0.0514	0.0408
$n=1000$	0.0861	0.0534	0.0504	0.0462
$n=2000$	0.0735	0.0575	0.0549	0.0463
$n=3000$	0.0772	0.0635	0.0528	0.0450

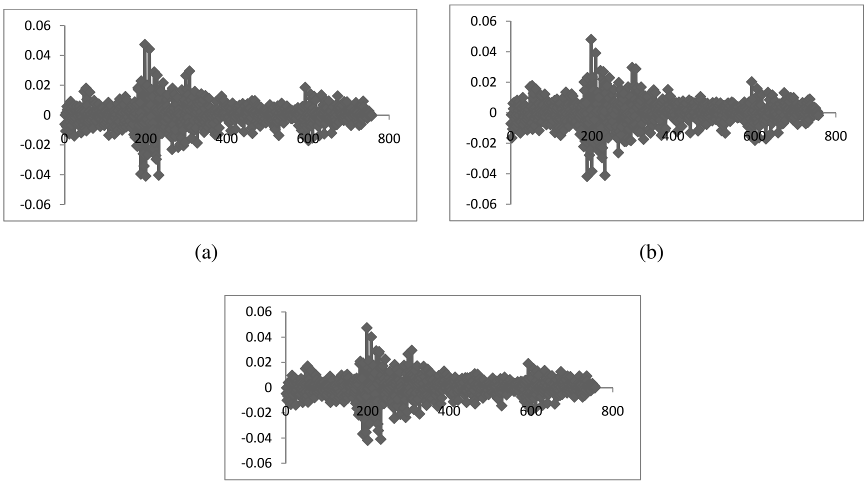
Table 2: Total variation distances of theoretical and empirical distributions (based on 50,000 simulations) of the time the process reaches ultimum maximum when $s=0.5$

distributed with density given by $f_{T_1}(t)=\frac{\delta}{\sqrt{2\pi}}exp(\delta\gamma)t^{-\frac{3}{2}}exp(-\frac{1}{2}(\delta^2t^{-1}+\gamma^2t)),t\geq 0$ Note that the location parameter in this model is zero and the parameter a , which represents the drift in the process $X_t^{\circ},t\geq 0$, is now the skewness parameter of the random variable X_1^0. To estimate the parameters a , s , δ and γ , we use the EM algorithm (Dempster et al., 1977). The EM algorithm is an iterative procedure that converges under rather weak conditions to a local maximum of the likelihood function. In this article, we shall only present the estimate values of the parameters and some interesting figures to show the tightness between the theoretical and empirical results. If one is interested to know more about the EM procedure, we refer the reader to Aas and Haff (2006), Karlis (2002) or Protassov (2004). The three sets of data we compiled and presented in Figure 1(a)-1(c) show the time series behavior and illustrate the jump component. They are daily logarithmic returns for the period from January 1, 2008 to December 31, 2010. If the jumps are suppressed, then the random time is locally deterministic. It has been observed (Barndorff-Nielsen and Shephard, 2001) that the volatility term is both stochastic and usually clustered, which is clearly indicated in Figures 1(a)-1(c). The clustering component appears to occur between September of 2008 to June of 2009 and April 2010 to August 2010 and this is consistent to all three indices.

To illustrate the goodness-of-fit of the three data sets, we provide QQ plots of the corresponding logarithmic returns against both Gaussian and the NIG distribution. If the comparisons are done against the Gaussian distribution, one can easily observe that both right and left tail are heavier than normal. However, if one compares the ordered data against the NIG scores, it can be seen that the data fit extremely well.

To illustrate the asymmetry of the shape of the data, we overlaid on the histogram the NIG distribution. It appears that the right tail is lighter than the left tail. It is also evident that the NIG distribution fits well in all three sets of the observed data.

The values of the estimates were a= -4.8978, s=1, δ= 0.0046 and γ= 66.1971 for the S&P 500, a= -4.1101, s=1, δ= 0.0056 and γ= 74.2608 for NASDAQ and a= -5.1098,



(a) (b)

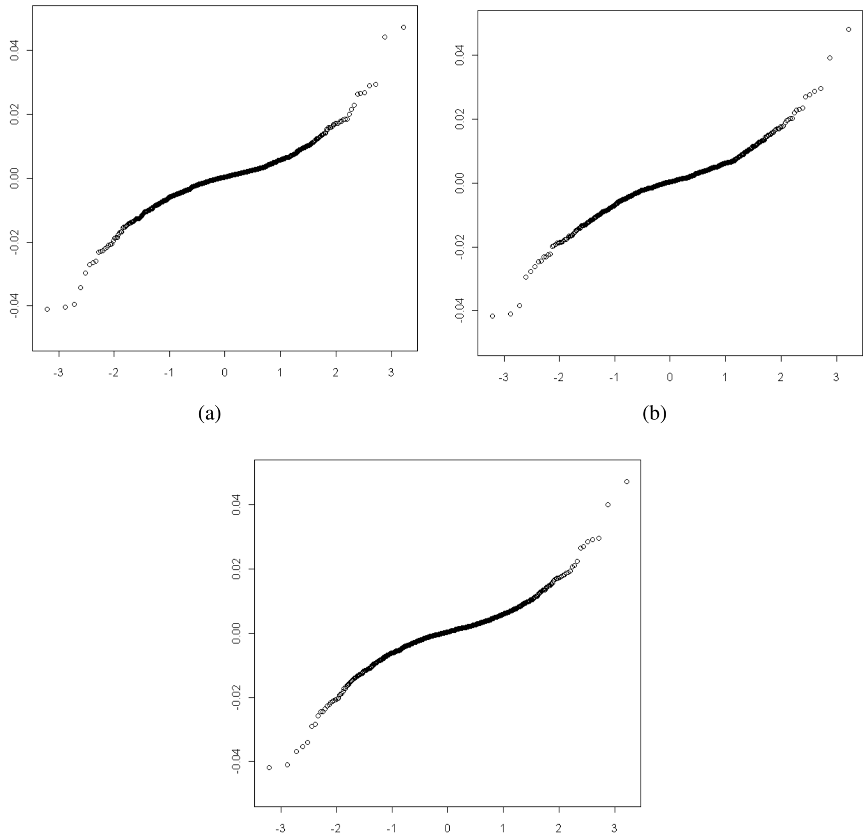
(c)

Figure 2: Time series plots of USA log-returns indices for the period from January 1, 2008 to December 31, 2010 for S&P 500 in (a), for NASDAQ in (b) and for NYSE in (c).

s=1, δ= 0.0049 and γ= 66.1993 , for NYSE. We stopped the EM procedure when the distance between successive iterations (distance of estimate values of parameters between successive iterations) was smaller than 10^{-10}.

Barndorff-Nielsen and Prause (2001) also identify two more measures that can play a significant role in recognizing the shape distribution of NIG, the steepness ξ and the asymmetry χ. These parameters are functions to the NIG model parameters and can be derived using the expressions $\xi = \{1+\delta(\alpha^2 - a^2)\}^{-\frac{1}{2}}, \chi = fraca\xi\alpha$, where the parameter α is determined by $\alpha = \{\gamma^2 + a^2\}^{\frac{1}{2}}$. Their estimate values for the specific data sets are: ξ=.8755 ,χ=-0.0646 , ξ=.8404 ,χ=-0.0464 and ξ=.8689 ,χ=-0.0669 , for S&P 500, NASDAQ and NYSE, respectively. It is also noted (Barndorff-Nielsen and Prause, 2001) and empirically verified that the estimate value of ξ for financial data lies in the interval [0.7, 0.9] . This is also true for the above sets.

To demonstrate the applicability of the density described in Theorem 9(4), we confine our data analysis to only the 2008 period. This is an event based approach, i.e., we consider a segment of time that an event occurs only in a specific location (USA). It is well documented (National Bureau of Economic Research) that from January of 2008 to the end of the same year the US economy was in recession. This is exactly the reason we consider our analysis only for the year of 2008. We found that the estimate parameter a for the year 2008 for the three indices S&P 500, NASDAQ and NYSE were -5.8472, -5.4182 and -6.8231, respectively. Note that the parameters for the subordinator are not required in this analysis. Using these values as parameters, we found that the 95% coverage probability were 2, 2, 1 days. It should be noted that the coverage probability is very small. In all three cases, the observed time at which the maximum occurred was at zero. This clearly confirms our findings.



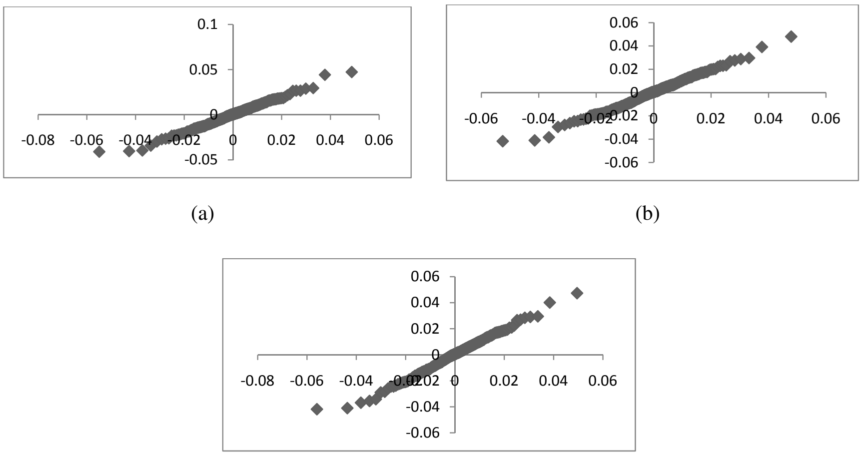
(a) (b) (c)

Figure 3: QQ plots of USA indices against normal distribution for S&P 500 in (a), for NASDAQ in (b) and for NYSE in (c).

In the preceding sections, we considered various statistical facts that emerged from empirical evidence studying financial prices and we proposed that the subordinated Gaussian process is an excellent candidate to model asset prices. Our empirical findings suggest that the recommended model performs extremely well. Contingent upon the fact that the subordinated process is the right scheme, we obtained a distribution for the time it takes a subordinated Gaussian process to reach its maximum. We applied this distribution to the USA indices for a recession period, which constitutes an event based period. Our data findings strongly support this specific distribution.

References

[1] AAS, K. AND HAFF, I.H. (2006). The generalized hyperbolic skew student's $t-$distribution. Journal of Financial Econometrics, 4, 275-309.



(a) (b)

(c)

Figure 4: QQ plots of USA indices against NIG distribution for S&P 500 in (a), for NASDAQ in (b) and for NYSE in (c).

[2] ABRAMOWITZ, M. AND STEGUN, I.A. (1970). Handbook of Mathematical Functions. Dover Publications, Inc., New York.

[3] BARNDORFF-NIELSEN, O.E. (1997). Normal inverse Gaussian distribution and the modeling of stock returns. Scandinavian Journal of Statistics, 24, 1-13.

[4] BARNDORFF-NIELSEN, O.E. AND SHEPHARD N. (2001). Non-Gaussian Ornstein-Uhlenbeck-based models and some of their uses in financial economics (with discussion). Journal of Royal Statistical Society, Series B, 63, 167-241.

[5] BARNDORFF-NIELSEN, O.E. AND PRAUSE, K. (2001). Apparent scaling. Finance and Stochastics, 5, 103-113.

[6] BERTOIN, J. (1996). Lévy processes. Cambridge Univ. Press, Melbourne, NY.

[7] BINGHAM, N.H. (1975). Fluctuation theory in continuous time. Advanced Applied Probability 7, 705-766.

[8] BLACK, F. AND SCHOLES, M. (1973). The pricing of options and corporate liabilities. The Journal of Political Economy, 81, 637-654.

[9] CARR, P., GEMAN, H., MADAN, D.B. AND YOR, M. (2001). Stochastic volatility for Lévy processes. Mathematical Finance, 13, 345-382.

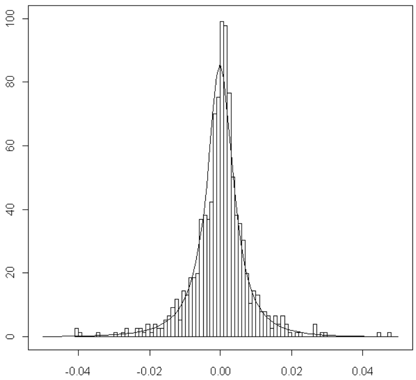
[10] CHUNG, K.L. AND DOOB, J.L. (1965). Fields, optionality and measurability. American Journal of Mathematics, 87, 397-424.

[11] CONT, R. (2001). Empirical properties of asset returns: stylized facts and statistical issues. Quantitative Finance, 1, 223-236.

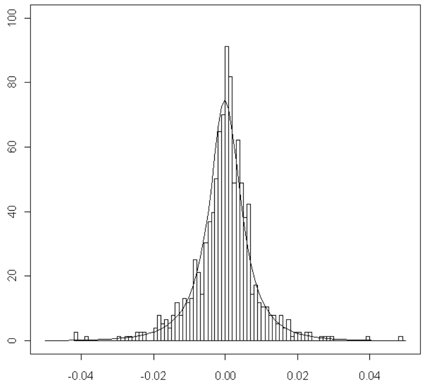
[12] DE VRIES, C.G. (1994). Stylized facts of nominal exchange rate returns. The Handbook of International Macroeconomics, 348-389, Blackwell.

[13] DEMPSTER, A.P., LAIRD, N.M., AND RUBIN, D.B. (1977). Maximum likelihood from incomplete data via the EM algorithm. Journal of the Royal Statistical Society. Series B, 39, 1-38.

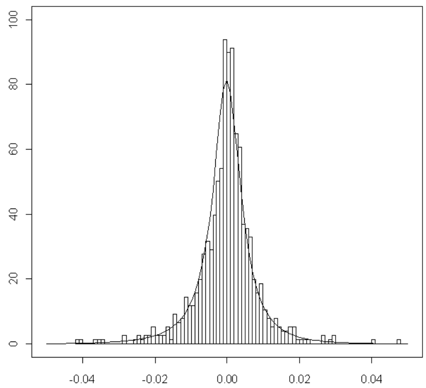
[14] DONEY, R.A. (2005). Fluctuation theory for Lévy processes. Springer & Verlag (in print).



(a)



(b)



(c)

Figure 5: Plots of NIG distribution and empirical distribution of USA indices for the period from January 1, 2008 to December 31, 2010 for S&P 500 in (a), for NASDAQ in (b) and for NYSE in (c).

[15] EBERLEIN, E., KELLER, U. AND PRAUSE, K. (1998). New insights into smile, mispricing and value at risk: the hyperbolic model. Journal of Business, 71, 371-405.

[16] FAMA, E.F. (1965). The behavior of stock-market prices. The Journal of Business, 38, 34-105.

[17] FELLER, W.R. (1971). An Introduction to Probability Theory and Its Applications, Vol. II. Wiley, New York. MR0270403.

[18] GEMAN, H., MADAN, D. AND YOR, M. (2001). Time changes for Lévy processes. Mathematical Finance, 11, 79-96.

[19] GREENWOOD, P.E. AND PITMAN, J.W. (1980). Fluctuation identities for Lévy processes and splitting at the maximum. Advanced of Applied Probability, 12, 893-902.

[20] KARATZAS, I. AND SHREVE, S.E. (1988). Brownian Motion and Stochastic Calculus. Springer- Verlag, New York.

[21] KARLIS D. (2002). An EM type algorithm for maximum likelihood estimation of the normalinverse Gaussian distribution. Statistics and Probability Letters, 57, 43-52.

[22] KYPRIANOU, A.E. (2004). Some calculations for Israeli options. Finance and Stochastics, 8, 73-86.

[23] MANDELBROT, B. (1963). The variation of certain speculative prices. The Journal of Business, 36, 394-419.

[24] MERTON, R. C. (1973). Theory of rational option pricing. The Bell Journal of Economics and Management Science Vol. 4 Issue 1 141-183.

[25] MULLER, U.A, DACOROGNA, AND M.M. PICKTET, O.V. (1998). Heavy tails in high-frequency financial data. A Practical Guide to Heavy Tails, Eds. Adler, R.J., Feldman, R.E. and Taqqu, 55-78, Birkhauser, Boston.

[26] PECHERSKII, E. A. AND ROGOZIN. B. A. (1969). On joint distributions of random variables associated with fluctuations of a process with independent increments. Theory of Probability and its Applications, 14, 410-423.

[27] PLEROU, V., GOPIKRISHNAN, P., AMARAL L.A.N., MEYER, M. AND STANLEY, H. E. (1999). Scaling of the distribution of price fluctuations of individual companies. Physical Review E., 60, 5305. 5316.

[28] PROTASSOV, R. S. (2004). EM-based maximum likelihood parameter estimation for multivariate generalized hyperbolic distributions with fixed λ. Statistics and Computing, 14, 67-77.

[29] ROGOZIN, B. A. (1966). On distributions of functions related to boundary problems for processes with independent increments. Theory of Probability and its Applications 11, 580-591.

[30] SAMUELSON, P. A. (1965). Rational theory of warrant pricing. Industrial Management Review, 6, 13-32.

[31] SATO, K-I. (1999). Lévy processes and infinitely divisible distributions. University Press, Cambridge.