The Fourth International Conference on Computational Systems Biology (ISB2010)
Suzhou, China, September 9–11, 2010

Stability Analysis of Nonlinear Hybrid System in Microbial Fed-batch Culture*

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Abstract In this paper, we study the stability of the nonlinear hybrid system describing the concentrations of extracellular and intracellular substances in the process of bio-dissimilation of glycerol to 1,3-propanediol. We prove an asymptotical stability lemma in nonlinear impulsive hybrid systems and obtain the strict stability and uniformly asymptotical stability for the nonlinear hybrid system. These results provide less conservative stability conditions for hybrid system as compared to classical results in the literature and allow us to characterize the invariance of a class of nonlinear hybrid dynamical systems.

Keywords Stability, Nonlinear hybrid system, Microbial fermentation.

1 Introduction

1,3-propanediol(1,3-PD) possesses potential applications on a large commercial scale, especially as a monomer of polyesters or polyurethanes, its microbial production is recently paid attention to in the world for its low cost, high production and no pollution, etc. Among all kinds of microbial production of 1,3-PD, dissimilation of glycerol to 1,3-PD by Klebsiella pneumonia has been widely investigated since 1980s due to its high productivity[1]. The experimental investigations showed that the fermentation of glycerol by K. pneumonia is a complex bioprocess, since the microbial growth is subjected to multiple inhibitions of substrate and products. Over the past several years, great progress has been made in studying the nonlinear dynamical system of continuous fermentation of glycerol by K. pneumonia, including the quantitative description of the kinetics of cell growth, substrate consumption and product formation [2,3], and so on. In researches on fed-batch culture, all numerical results are based on the continuous dynamical models and there exist big errors between computational and experimental results. In fact, there exist impulsive phenomena in fed-batch culture, so the process characterized by continuous models is not fit for the actual process any longer. In order to characterize the actual process, the impulsive differential equations are applied to the fed-batch fermentation[4]. In recent years, nonlinear impulsive, multistage and hybrid dynamical systems have been explored to formulate the fed-batch culture with coupled open loop inputs of glycerol and alkali [5]. The parameters in continuous system are not fit for the impulsive system, so

*This work was supported by the Natural Science Foundation of Inner Mongolia (grant no. 2009MS0104) and 513 Program' Project of Inner Mongolia University.
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parameter identification is necessary. Usually, ranges of parameters change in the neighborhood of initial values during the identification process. But we can't ensure the system is stable under the given ranges of parameters. Thus, stability of the system becomes a fundamental issue in system analysis and design, that is necessary for system identification and optimal control.

In this paper, we consider the impulsive dynamical system in [6] as preliminaries. The paper is organized as follows. In section 2 a nonlinear hybrid system is described for the fed-batch culture. Section 3 analyzes the strict stability and uniformly asymptotical stability for the model. Discussions and conclusions are presented at the end of this paper.

2 Nonlinear hybrid system in fed-batch culture

The fed-batch culture begins with batch fermentation, then batch-fed glycerol and alkali are discontinuously added to the reactor every so often in order that glycerol concentration keeps in a proper range and the pH of the solution in a required level.

According to the factual experiments, we make the following assumptions.

(A1) The concentration of reactants are uniform in reactor, while time delay and nonuniform space are ignored.

(A2) the sub substrates added to the reactor only include glycerol and alkali. Under assumptions (A1) and (A2), the fed-batch process can be formulated by

$$\frac{dC_X(t)}{dt} = (\mu - d)C_X(t) - \frac{F_G + F_N}{V(t)}C_X(t), \tag{1}$$

$$\frac{dC_{Gly}(t)}{dt} = -q_{Gly}C_X(t) + \frac{F_G}{V(t)}(C_{s_0} - C_{Gly}(t)) - \frac{F_N}{V(t)}C_{Gly}(t), \tag{2}$$

$$\frac{dC_{PD}(t)}{dt} = q_{PD}C_X(t) - \frac{F_G + F_N}{V(t)}C_{PD}(t), \tag{3}$$

$$\frac{dC_{HAc}(t)}{dt} = q_{HAc}C_X(t) - \frac{F_G + F_N}{V(t)}C_{HAc}(t), \tag{4}$$

$$\frac{dC_{EtOH}(t)}{dt} = q_{EtOH}C_X(t) - \frac{F_G + F_N}{V(t)}C_{EtOH}(t), \tag{5}$$

$$\frac{dC_{Na^+}(t)}{dt} = -\frac{F_G}{V(t)}C_{Na^+}(t) + \frac{F_N}{V(t)}(\rho - C_{Na^+}(t)), \tag{6}$$

$$\frac{dV(t)}{dt} = F_G + F_N, \tag{7}$$

where $C_X(t), C_{Gly}(t), C_{PD}(t), C_{EtOH}(t)$ are the concentrations of biomass, glycerol, 1,3-PD and ethanol in reactor at time t; $C_{HAc}(t)$ is the total concentration of acetic acid in reactor, including Ac^- ions, and $V(t)$ is the volume of the solution. If there is no confusion, we shall simplify $C_i(t)$ as $C_i, i = X, Gly, PD, HAc, EtOH, Na^+$, and $V(t)$ as V. d is the specific decay rate of cells [7]. $C_{s_0}(mmol/L)$ and $\rho(mmol/L)$ are the concentrations of glycerol and $NaOH$ in feed medium,
respectively. F_G and F_N are feeding velocities of glycerol and alkali, which are discrete variables respectively taking values from finite sets $S_1 := [0, v_1]$ and $S_2 := [0, v_2]$, where v_1 and v_2 are constant flow rates of glycerol and alkali pumps.

The specific growth rate of cells μ, specific consumption rate of substrate q_{Gly} and specific formation rate of products $q_i i = X, Gly, PD, HAc, EtOH$, are expressed by the following equations based on previous work [2].

$$\mu = \mu_m \frac{C_{Gly}}{C_{Gly}+k_s}(1-\frac{C_{Gly}}{C^*_{Gly}})(1-\frac{C_{PD}}{C^*_{PD}})(1-\frac{C_{HAc}}{C^*_{HAc}})(1-\frac{C_{EtOH}}{C^*_{EtOH}}), \tag{8}$$

$$q_{Gly} = m_2 + \frac{\mu}{Y_2} + \Delta_2 \frac{C_{Gly}}{C_{Gly}+k^*_2}, \tag{9}$$

$$q_{PD} = m_3 + \mu Y_3 + \Delta_3 \frac{C_{Gly}}{C_{Gly}+k^*_3}, \tag{10}$$

$$q_H Ac = m_4 + \mu Y_4 + \Delta_4 \frac{C_G ly}{C_G ly+k^*_4}, \tag{11}$$

$$q_{EtOH} = m_5 + \mu Y_5 + \Delta_5 \frac{C_G ly}{C_G ly+k^*_5}, \tag{12}$$

Here m_i, Y_i, Δ_i and $k^*_i, i = 2,3,4,5$, are parameters. μ_m is the maximum specific growth rate and k_s is a Monod saturation constant. The critical concentrations of glycerol, 1,3-PD, acetic acid and ethanol for cell growth are $C^*_{Gly} = 2039 mmol/L$, $C^*_{PD} = 1300 mmol/L$, $C^*_{HAc} = 1026 mmol/L$ and $C^*_{EtOH} = 360.9 mmol/L$, respectively. And the parameters in the model come from [6].

3 The stability analysis of the model

The nonlinear hybrid system $(1)-(7)$ can be converted into another identical nonlinear hybrid system (14) by using (7) of nonlinear hybrid system.

$$V(t) = (F_G + F_N)t + C.$$

Let $C = V_0$ and $V_0 = k(F_G + F_N)$, then

$$V(t) = (F_G + F_N)t + k(F_G + F_N). \tag{13}$$

Using (13), we may obtain the following nonlinear hybrid system:

$$\begin{cases} \frac{dC_X(t)}{dt} = (\mu - d)C_X(t) - \frac{1}{t+k}C_X(t), \\ \frac{dC_{Gly}(t)}{dt} = -q_{Gly}C_X(t) - \frac{1}{t+k}C_{Gly}(t) + \frac{F_N}{(t+k)(F_G+F_N)}C_{s_0}, \\ \frac{dC_{PD}(t)}{dt} = q_{PD}C_X(t) - \frac{1}{t+k}C_{PD}(t), \\ \frac{dC_{HAc}(t)}{dt} = q_{HAc}C_X(t) - \frac{1}{t+k}C_{HAc}(t), \\ \frac{dC_{EtOH}(t)}{dt} = q_{EtOH}C_X(t) - \frac{1}{t+k}C_{EtOH}(t), \\ \frac{dC_{Na^+}(t)}{dt} = -\frac{1}{t+k}C_{Na^+}(t) + \frac{F_N}{(t+k)(F_G+F_N)}\rho, \end{cases} \tag{14}$$

The equilibrium point of this system is $(0, \frac{F_N}{F_G+F_N}C_{s_0}, 0, 0, 0, \frac{F_N}{F_G+F_N}\rho)$. If we move this equilibrium point to $(0,0,0,0,0,0)$, the nonlinear hybrid system (14) will be changed into the following nonlinear hybrid system (15) correspondingly.

$$\begin{cases} \frac{dx_1(t)}{dt} = (\mu - d)x_1(t) - \frac{1}{t+k}x_1(t), \\ \frac{dx_2(t)}{dt} = -q_2 x_1(t) - \frac{1}{t+k}x_2(t), \\ \frac{dx_3(t)}{dt} = q_3 x_1(t) - \frac{1}{t+k}x_3(t), \\ \frac{dx_4(t)}{dt} = q_4 x_1(t) - \frac{1}{t+k}x_4(t), \\ \frac{dx_5(t)}{dt} = q_5 x_1(t) - \frac{1}{t+k}x_5(t), \\ \frac{dx_6(t)}{dt} = -\frac{1}{t+k}x_6(t), \end{cases} \tag{15}$$

For convenience here, we denote $C_X(t), C_{Gly}(t), C_{PD}(t), C_{HAc}(t), C_{EtOH}, C_{Na^+}, q_{Gly}, q_{PD}, q_{HAc}, q_{EtOH}$ as $x_1, x_2, x_3, x_4, x_5, x_6, q_2, q_3, q_4, q_5$ respectively.

In order to analyze the stability of the system. we consider the impulsive differential system in a real n-dimensional Euclidean space with norm $\|\cdot\|$.

$$\begin{cases} x' = f(t,x), & t \neq t_k \\ \Delta x = I_k(x), & t = t_k \\ x(t_0^+) = x_0, & t_0 \geq 0, k = 1, 2, \cdots, \end{cases} \tag{16}$$

Under the following assumption:

(H_1) $0 < t_1 < t_2 < \cdots < t_k < \cdots$, and $t_k \to \infty$ as $k \to \infty$;

(H_2) $f : R^+ \times R^n \to R^n$ is continuous in $(t_{k-1}, t_k] \times R^n$ and for each $x \in R^n$, $k = 1, 2, \cdots$, $\lim_{(t,y)\to(t_k^+,x)} f(t,y) = f(t_k^+, x)$ exists;

$(H_3) I_k : R^n \to R^n$.

Assume that $f(t,0) \equiv 0$ and $I_k(0) = 0$ for all k so that the trivial solution of (16) exists. Denote $S(\rho) = \{x \in R^n : \|x\| < \rho\}$. We define the following classes of function spaces:

$K = \{a \in C[R^+, R^+] : a$ is strictly increasing and $a(0) = 0\}$.

V_0={$V : R^+ \times R^n \to R^+$čž$V$ is continuous in $(t_{k-1}, t_k] \times R^n$ and for each $x \in R^n, k = 1, 2, \cdots, \lim_{(t,y)\to(t_k^+,x)} V(t,y) = V(t_k^+, x)$ exists; V is locally Lipschitz in x }.

For $V \in V_0, (t,x) \in (t_{k-1}, t_k] \times R^n$, define the generalized derivatives of the dynamical system (16).

$$D^+V(t,x) = \limsup_{h\to 0^+} \frac{1}{h}[V(t+h, x+hf(t,x)) - V(t,x)],$$

$$D_-V(t,x) = \liminf_{h\to 0^-} \frac{1}{h}[V(t+h, x+hf(t,x)) - V(t,x)].$$

Definition 1. The trivial solution of the system (16) is said to be
(S_1) practically stable, if given (λ, A) with $0 < \lambda < A$, we have $\|x_0\| < \lambda$ implies $\|x(t)\| < A$, $t \geq t_0$ for some $t_0 \in R^+$;
(S_2) uniformly practically stable, if (S_1) holds for all $t_0 \in R^+$;
(S_3) strict practically stable, if (S_1) holds and for every $\mu \leq \lambda$, there exists $B < \mu$ such that $\|x_0\| > \mu$ implies $\|x(t)\| > B$, $t \geq t_0$;
(S_4) strict uniformly practically stable, if (S_3) holds for all $t_0 \in R^+$;
(S_5) uniformly attractive, if given $\delta > 0, \varepsilon > 0$, there exists $T = T(\varepsilon)$, such that $\|x_0\| < \delta$ implies $\|x(t)\| < B$, $t \geq t_0 + T$;
(S_6) uniformly practically asymptotically stable, if (S_2) ,(S_5) hold with $\delta = \lambda$.

Lemma 1. *(See*[8]*) Let $V \in V_0$ and suppose that*

$$\begin{cases} D^+V(t,x) \leq \ g(t,V(t,x)), & t \neq t_k \\ V(t^+, x+I_k(x)) \leq \psi_k(V(t,x)), & t = t_k \end{cases}$$

where $g : R^+ \times R^+ \to R$, g satisfies (H_2) and $\psi_k : R^+ \to R^+$ is nondecreasing. Let $r(t) = r(t,t_0,u_0)$ be the maximal solution of the comparison system of

$$\begin{cases} u' = g(t,u), & t \neq t_k \\ u(t_k^+) = \psi_k(u(t_k)), & t = t_k \\ u(t_0^+) = u_0 \geq 0, & k = 1,2,\cdots, \end{cases}$$

existing on $[t_0, \infty)$. Then $V(t_0^+, x_0) \leq u_0$ implies that $V(t,x(t)) \leq r(t), t \geq t_0$, where $x(t) = x(t,t_0,x_0)$ is any solution of (16) *existing on $[t_0, \infty)$.*

Lemma 2. *Suppose that*
(i) $0 < \lambda < A < \rho$;
(ii) *There exist $V : R^+ \times S(\rho) \to R^+$, $V_1 \in V_0$, $a_1, b_1 \in K$ for $(t,x) \in R^+ \times S(\rho)$, such that*

$$b(\|x\|) \leq V(t,x) \leq a(\|x\|)$$

and

$$\begin{cases} D^+V(t,x) \leq \ g(t,V(t,x)), & t \neq t_k \\ V(t, x+I_k(x)) \leq \psi_k(V(t,x)), & t = t_k \end{cases}$$

where $g : R^+ \times R^+ \to R$, $g(t,0) \equiv 0$ and g satisfies (H_2). $\psi_k : R^+ \to R^+$ is nondecreasing;
(iii) *The comparison system*

$$\begin{cases} u' = \ g(t,u), & t \neq t_k \\ u(t_k^+) = \psi_k u(t_k)), & t = t_k \\ u(t_0^+) = u_0 \geq 0, & k = 1,2,\cdots, \end{cases}$$

is uniformly practically asymptotically stable with respect to some (λ_1, A_1);
(iv) $a(\lambda) \leq b(A)$;
Then the trivial solution of (16) *is uniformly practically asymptotically stable.*

Proof. We claim that if $\|x_0\| < \lambda$ we have $\|x(t)\| < \lambda$, $t \geq t_0$ where $x(t) = x(t, t_0, x_0)$ is any solution of (16). If it is not true, there exists a solution $x(t) = x(t, t_0, x_0)$ of (16) with $\|x(t)\| < \lambda$ and $t^2 > t^1 > t_0, t^2, t^1 \neq t_k$ such that

$$\|x(t^1)\| < \lambda, \|x(t^2)\| \geq A$$

Hence we get by Lemma 1 using condition (ii)

$$V(t,x) \leq r(t, t_0, V(t_0^+, x_0)), t_0 \leq t \leq t^2, \tag{17}$$

where $r(t, t_0, V(t_0^+, x_0))$ is the maximal solution of the comparison system through $(t_0, V(t_0^+, x_0))$. Combining condition (ii) and (17), we have

$$V(t^2, x(t^2)) \leq r(t^2, t^1, V(t^1, x(t^1))) < a(\lambda) \leq b(A)$$

But by condition(ii), we obtain

$$b(A) \leq b(\|x(t^2)\|) \leq V(t^2, x(t^2))$$

This is a contradiction, so the claim is valid.

To complete the proof, we need prove the trivial solution of (16) is uniformly attractive. Since $u \equiv 0$ of the comparison system is uniformly attractive, given $b(\varepsilon) > 0$, there exist a $T = T(\varepsilon)$ such that $u_0 < \lambda_1$ implies

$$u(t) < b(\varepsilon), t \geq t_0 + T$$

Defining $\lambda^* = \min(\lambda, a^{-1}(\lambda_1))$, we choose $\|x_0\| < \lambda^*$, then $V(t_0^+, x_0) \leq a(\|x_0\|) < a(\lambda^*) \leq \lambda_1$. It follows that

$$b(\|x(t)\|) \leq V(t, x(t)) \leq r(t, t_0, V(t_0^+, x_0)) < b(\varepsilon), t \geq t_0 + T,$$

which proves that the trivial solution of (16) is uniformly attractive. The proof is completed.

Lemma 3. *(See*[8]*). Suppose that*
(i) $0 < \lambda < A < \rho$;
(ii) *There exists* $V_1 : R^+ \times S(\rho) \to R^+, V_1 \in V_0, a_1, b_1 \in K$ *such that* $a_1(\lambda) \leq b_1(A)$ *for* $(t,x) \in R^+ \times S(\rho)$,

$$b_1(\|x\|) \leq V_1(t,x) \leq a_1(\|x\|)$$

and

$$\begin{cases} D^+V_1(t,x) \leq \ 0, & t \neq t_k \\ V_1(t^+, x + I_k(x)) \leq (V_1(t,x)), & t = t_k \end{cases}$$

where $g_1 : R^+ \times R^+ \to R, g_1(t,0) \equiv 0$ *and* g_1 *satisfies* (H_2).
(iii) *There exists* $V_2 R^+ \times S(\rho) \to R^+ V_2 \in V_0, a_2, b_2 \in K$*such that for* $(t,x) \in R^+ \times S(\rho)$,

$$b_2(\|x\|) \leq V_2(t,x) \leq a_2(\|x\|)$$

and

$$\begin{cases} D_- V_2(t,x) \geq 0, & t \neq t_k \\ V_2(t^+, x + I_k(x)) \geq (V_2(t,x)), & t = t_k \end{cases}$$

where $g_2 : R^+ \times R^+ \to R$, $g_2(t,0) \equiv 0$ *and* g_2 *satisfies* (H_2). *Then the trivial solution of* (16) *is uniformly strictly practically stable.*

Theorem 1. *The equilibrium point of the nonlinear hybrid system* (15) *is uniformly practically asymptotically stable.*

Proof. The proof of the theorem can be decomposed three parts. Firstly, we construct function

$$a(x) = 8x^2, b(x) = \frac{1}{2}x^2, V = \frac{(x_1+2x_2+x_3+x_4+x_5+x_6)^2}{2}$$

and prove that

$$b(\|x\|) \le V(x) \le a(\|x\|).$$

Since

$a(\|x\|) - V(x) = 8(x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+x_6^2) - \frac{(x_1+2x_2+x_3+x_4+x_5+x_6)^2}{2}$
$= \frac{15}{2}(x_1^2+x_3^2+x_4^2+x_5^2+x_6^2) + \frac{12}{2}x_2^2 - x_1(x_3+x_4+x_5+x_6) - 2x^2(x_3+x_4+x_5+x_6)$
$- x_3(x_4+x_5+x_6) - x_4(x_5+x_6) - x_5x_6$
$\ge \frac{15}{2}(x_1^2+x_3^2+x_4^2+x_5^2+x_6^2) + \frac{12}{2}x_2^2 - x_1 \max_{3\le i\le 6}\{x_i\} - 2x_2 \max_{3\le i\le 6}\{x_i\} - x_3 \max_{4\le i\le 6}\{x_i\} - x_4 \max_{5\le i\le 6}\{x_i\} -$
x_5x_6
$\ge \frac{15}{2}(x_1^2+x_3^2+x_4^2+x_5^2+x_6^2) + \frac{12}{2}x_2^2 - (x_1+2x_2+x_3+x_4+x_5)\max_{3\le i\le 6}\{x_i\}$
$\ge \frac{15}{2}(x_1^2+x_3^2+x_4^2+x_5^2+x_6^2) + \frac{12}{2}x_2^2 - \frac{12}{2}\max_{3\le i\le 6}\{{x_i}^2\}$
≥ 0

we have

$$a(\|x\|) \ge V(x).$$

Since

$$V(x) - b(\|x\|) = \frac{(x_1+2x_2+x_3+x_4+x_5+x_6)^2}{2} - \frac{x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+x_6^2}{2} \ge 0$$

we have

$$V(x) \ge b(\|x\|).$$

Secondly, we complete the proof of condition (ii) of Lemma 2. We have

$D^+V(x) = (x_1+2x_2+x_3+x_4+x_5+x_6)(x_1'+2x_2'+x_3'+x_4'+x_5'+x_6')$
$= (x_1+2x_2+x_3+x_4+x_5+x_6)[(\mu-d-\frac{1}{t+k})x_1 + (-2q_2x_1 - \frac{2}{t+k}x_2) + (q_3x_3 - \frac{1}{t+k}x_3)$
$\quad + q_4x_1 - \frac{1}{t+k}x_4) + (q_5x_1 - \frac{1}{t+k}x_5) - \frac{1}{t+k}x_6]$
$= (x_1+2x_2+x_3+x_4+x_5+x_6)[(\mu-d-2q_2+q_3+q_4+q_5)x_1 - \frac{1}{t+k}(x_1+2x_2+x_3+x_4+$
$x_5+x_6)]$

Let

$\xi = \mu-d-2q_2+q_3+q_4+q_5 = \mu-d-2(m_2+\frac{\mu_m}{Y_2}+\Delta_2\frac{x_2}{x_2+k_2^*}) + (m_3+\mu Y_3+\Delta_3\frac{x_2}{x_2+k_3^*})$
$\quad + (m_4+\mu_m Y_4+\Delta_4\frac{x_2}{x_2+k_4^*}) + (m_5+\mu_m Y_5+\Delta_5\frac{x_2}{x_2+k_5^*}).$

we have

$\xi = (m_3+m_4+m_5-2m_2-d) + \mu(1+Y_3+Y_4+Y_5-\frac{2}{Y_2}) + (\Delta_3\frac{x_2}{x_2+k_3^*}+\Delta_4\frac{x_2}{x_2+k_4^*}+\Delta_5\frac{x_2}{x_2+k_5^*}-$
$2\Delta_2\frac{x_2}{x_2+k_2^*})$
$\le (-2.6703+0.00292+0.00408-0-0.025) + \mu(1+97.7859+15.3834+10.7331-$
$\frac{2}{0.0131})$
$+[(7.5597+0.0104+0.1022)\times\frac{x_2}{x_2+110} - 12.0139\times\frac{2x_2}{x_2+90}]$

≤ 0
So we can get

$$D^+V(x) \leq -\frac{1}{t+k}(x_1+2x_2+x_3+x_4+x_5+x_6)^2 = -\frac{2}{t+k}V(x).$$

Next we will prove

$$V(x(t_k)+I_k(x(t_k))) \leq V(x(t_k)), for \quad t \neq t_k$$

Since $\Delta x = I_k(x) = 0$, for $t \neq t_k$,
then

$$V(x(t_k)+I_k(x(t_k))) = V(x(t_k))$$

So

$$\begin{cases} D^+V(x) \leq -\frac{2}{t+k}V(x), & t \neq t_k \\ V(x(t_k)+I_k(x(t_k))) = V(x(t_k)), & t = t_k \end{cases}$$

Further, we can get the comparison system

$$\begin{cases} u' = -\frac{2}{t+k}u, & t \neq t_k \\ u(t_k^+) = u(t_k)), \\ u(t_0^+) = u_0 \geq 0 & k = 1,2,\cdots, \end{cases}$$

Thirdly, we will prove that u is uniformly practically asymptotically stable with respect to some (λ_1, A_1). Using the comparison system, we get $u(t) = \frac{u_0}{(t+k)^2}$. This implies that for $\lambda_1 < A_1$, $u_0 < \lambda_1$

$$u(t) = \frac{u_0}{(t+k)^2} \leq \frac{\lambda_1}{(t+k)^2} \leq \lambda_1 < A_1$$

To complete the proof, we need prove u is uniformly attractive. If given $\delta > 0, \varepsilon > 0$ and $\delta < \varepsilon$, such that $u_0 < \delta$ implies $u(t) = \frac{u_0}{(t+k)^2} < \frac{\delta}{(t+k)^2} < \varepsilon$, only need $t < \sqrt{\frac{\varepsilon}{\delta}} - 1$ then given $\delta > 0, \varepsilon > 0$ and $\delta < \varepsilon$, there exists $T = \sqrt{\frac{\varepsilon}{\delta}} - 1 - t_0$, such that $u_0 < \delta$ implies $u(t) < \varepsilon$, for $t \geq t_0 + T$.

Combining above-mentioned three parts, by Lemma 2 the proof of the theorem is completed.

Theorem 2. *The equilibrium point of the nonlinear hybrid system* (15) *is uniformly strict practically stable.*

Proof. The theorem is proved by four parts. Firstly, we construct function

$$a_1(x) = 8x^2, b_1(x) = \frac{1}{2}x^2, V_1 = \frac{(x_1+2x_2+x_3+x_4+x_5+x_6)^2}{2},$$

by the proof of Theorem 1, we can obtain

$$b_1(\|x\|) \leq V_1(x) \leq a_1(\|x\|)$$

Secondly, by the proof of Theorem 1, we can obtain

$$D^+V_1(x) \leq -\frac{2}{t+k}V_1(x) \leq 0, V_1(x(t_k)+I_k(x(t_k))) = V_1(x(t_k))$$

So

$$\begin{cases} D^+V_1(x) \leq 0, & t \neq t_k \\ V_1(x(t_k)+I_k(x(t_k))) = V_1(x(t_k)), & t = t_k \end{cases}$$

Thirdly, we construct function

$$a_2(x) = \exp\{-\frac{1}{2}x^2\}, b_2(x) = \exp\{-8x^2\}, V_2 = \exp\{-\frac{(x_1+2x_2+x_3+x_4+x_5+x_6)^2}{2}\},$$

Next we need prove that

$$b_2(\|x\|) \leq V_2(x) \leq a_2(\|x\|).$$

Since

$$\frac{x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+x_6^2}{2} \leq \frac{(x_1+2x_2+x_3+x_4+x_5+x_6)^2}{2} \leq 8(x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+x_6^2),$$

then

$$\exp\{-8x^2\} \leq \exp\{-\frac{(x_1+2x_2+x_3+x_4+x_5+x_6)^2}{2}\} \leq \exp\{-\frac{1}{2}x^2\}.$$

So

$$b_2(\|x\|) \leq V_2(x) \leq a_2(\|x\|).$$

Fourthly, by the proof of Theorem 1, we can obtain

$$\begin{aligned} &D_-V_2(x) \\ &= -2(x_1+2x_2+x_3+x_4+x_5+x_6)^2)(x_1'+2x_2'+x_3'+x_4'+x_5'+x_6')\exp\frac{(x_1+2x_2+x_3+x_4+x_5+x_6)^2}{2} \\ &\geq \frac{4}{t+1}V_1(x)V_2(x) \geq 0,\ V_2(x(t_k)+I_k(x(t_k))) = V_2(x(t_k)). \end{aligned}$$

So

$$\begin{cases} D_-V_2(x) \geq 0, & t \neq t_k \\ V_2(x(t_k)+I_k(x(t_k))) = V_2(x(t_k)), & t = t_k \end{cases}$$

Combining above-mentioned four parts, by Lemma 3 the proof of the theorem is completed.

4 Conclusions

This paper develops an asymptotical stability lemma in nonlinear impulsive hybrid systems and analyzes the stability of nonlinear impulsive hybrid systems in microbial fed-batch culture by using strict stability theorem and uniformly asymptotical stability theorem. It shows that the system has strict stability and uniformly asymptotical stability.

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