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# Proposition of New Network DEA Models Based on the Unified DEA Model

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**Abstract:** In order to overcome the shortcomings of traditional DEA models respect to ratio measures, reference sets, and definitions of distance, we previously proposed a unified DEA model that included super-efficiency measures. In the present paper, we applied the proposed model to evaluating efficiencies of prefectures and discuss the usefulness of the unified DEA model. We have defined prefectures as matrix-type organizations and analyzed the prefecture efficiencies, using new network DEA models that maximize slacks subject to constraints including link constraints between sectors. Maximization of slacks is contrary to the philosophy of the unified DEA model. Link constraints may be too severe. Based on the philosophy of the unified DEA model, we discuss the possibility of new network DEA models.

**Keywords**: network DEA, reference set, efficient frontier, unified DEA

## 1 Introduction

Minimization of the objective function in the SBM (Slacks-Based Measure) model [1] results in the maximization of the slacks sum. This maximization corresponds to finding a point in the production possibility set that is the farthest point from each Decision Making Unit (DMU) to be evaluated, that is, the most difficult point to reach.

The SBM model maximizes distances, but the SuperSBM model minimizes distances. This super-SBM model has a shortcoming, discontinuities in the resulting efficiency scores.

To overcome these shortcomings, we proposed the unified DEA model [2],[3].

In the present paper we first apply the unified DEA model to evaluate efficiencies of prefectures and analyze its usefulness. We defined prefectures as matrix-type organizations and analyzed their efficiencies, using new network DEA models that maximize slacks subject to constraints including link constraints between sectors. However maximization of slacks is contrary to the philosophy of the unified DEA model. Link constraints may be too severe. Based on the philosophy of the unified DEA model, we propose a new network DEA model which relaxes link constraints.

## 2 Unified DEA Model

We proposed the unified DEA model including super-efficiency measure, based on the following principle:

**[Principle]** After deriving the nearest point, *Z*, to the target DMU *o* among points on the efficient frontier in $P \setminus (\mathbf{x}_o,\mathbf{y}_o)\}$, the production possibility set spanned by (X,Y) excluding $(\mathbf{x}_o,\mathbf{y}_o)$, obtain the efficiency score, using slacks between DMU *o* and the point *Z*, where

$$P \setminus (\mathbf{x}_o, \mathbf{y}_o) = \left\{ (\bar{\mathbf{x}}, \bar{\mathbf{y}}) \mid \bar{\mathbf{x}} \geq \sum_{j=1,\neq o}^{N} \lambda_j x_j,\ \bar{\mathbf{y}} \leq \sum_{j=1,\neq o}^{N} \lambda_j y_j,\ \bar{\mathbf{y}} \geq \mathbf{0},\ \boldsymbol{\lambda} \geq \mathbf{0} \right\},$$

where $(\mathbf{x}_j,\mathbf{y}_j)$ is inputs and outputs of DMU *j*.

Concrete procedures are as follows:

**<Phase 1>** Derivation of efficient facets

(i) After deleting DMU *o*, derive a set of $n_0$ efficient DMUs , *EF*, in the additive model. Let $K = 0$, $C = n_0 - 1$, and $E_0 = \phi$.

(ii) Make the combinations of *C* efficient DMUs: $EC_1$, $EC_2$, … , $EC_D$, where *D* is the number of ways that *C* DMUs can be chosen from $n_0$ efficient DMUs. Let $j = 1$.

(iii) If $EC_j \not\subset E_h\ (h \leq K)$ and a point corresponding to the centroid of $EC_j$ is efficient in the additive model, let the $EC_j$ be an efficient facet, $E_{K+1}$. Replace *K* with *K*+1 and *j* with *j*+1. If $j \leq D$, then repeat (iii). Otherwise replace *C* with *C*–1 and go to (iv).

(iv) If $C = 1$, then all of the efficient facets, $E_k$ $(k = 1,\ldots,K)$, are obtained. Otherwise return to (ii).

**<Phase 2>** Decision of the nearest point on effient facets to DMU *o*

(i) Determine the nearest point, $Z_k$, to DMU *o* on the efficient facet, $E_k$ $(k = 1,\ldots,K)$, where

$$D_{op(k)} = \frac{1}{2}\left\{ \frac{1}{M}\left( \sum_{i=1}^{M} \frac{|x_{io} - x_{ip(k)}|}{x_{io}} \right) + \frac{1}{R}\left( \sum_{j=1}^{R} \frac{|y_{jo} - y_{jp(k)}|}{y_{jo}} \right) \right\}$$

$$x_{ip(k)} = \sum_{h \in E_k} \lambda_h x_{ih},\quad y_{jp(k)} = \sum_{h \in E_k} \lambda_h y_{jh},$$

$$D_{oZ_k} = \min_{\lambda_h : h \in E_k} D_{op(k)}.$$

(ii) Decide the nearest point, *Z*, to DMU *o* on the efficient frontier, that is, $D_{oZ} = \min_k D_{oZ_k}$.

**<Phase 3>** Calculation of efficiency scores, $\eta_1$, $\eta_2$ and $\eta_3$, where

$$\eta_1 = (1 - \omega_x)/(1 + \omega_y)$$

$$\eta_2 = 1 - (\omega_x + \omega_y)/2$$

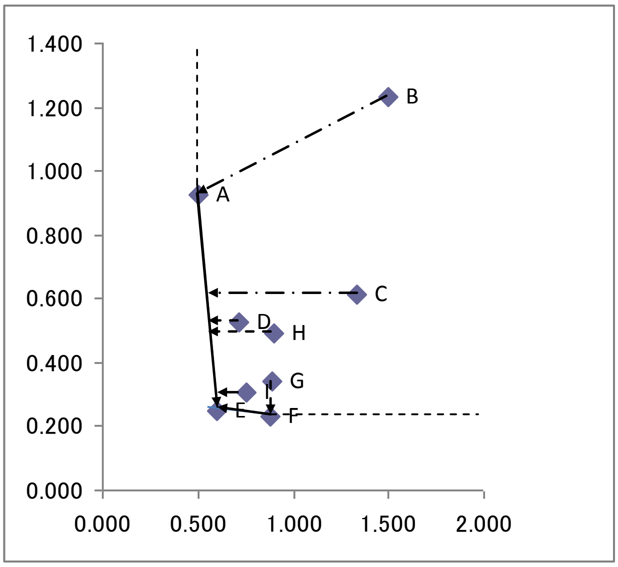


Fig. 1 An example of reference points

Table 1 SBM efficiency scores, 2

| DMU | SBMscore | DMU | SBMscore | DMU | SBMscore |
|---|---|---|---|---|---|
| **1** | **1.000** | 21 | 0.958 | 41 | 0.992 |
| 2 | 0.924 | **22** | **1.000** | 42 | 0.989 |
| 3 | 0.945 | 23 | 0.991 | 43 | 0.962 |
| 4 | 0.998 | **24** | **1.000** | 44 | 0.970 |
| 5 | 0.978 | 25 | 0.975 | 45 | 0.927 |
| 6 | 0.941 | 26 | 0.990 | 46 | 0.960 |
| 7 | 0.984 | **27** | **1.000** | **47** | **1.000** |
| 8 | 0.986 | 28 | 0.985 | 48 | 0.963 |
| **9** | **1.000** | **29** | **1.000** | **49** | **1.000** |
| **10** | **1.000** | **30** | **1.000** | **50** | **1.000** |
| **11** | **1.000** | **31** | **1.000** | **51** | **1.000** |
| **12** | **1.000** | 32 | 0.958 | **52** | **1.000** |
| **13** | **1.000** | 33 | 0.987 | 53 | 0.971 |
| **14** | **1.000** | 34 | 0.965 | **54** | **1.000** |
| 15 | 0.946 | 35 | 0.995 | | |
| 16 | 0.959 | 36 | 0.986 | | |
| 17 | 0.947 | 37 | 0.991 | | |
| **18** | **1.000** | 38 | 0.983 | | |
| **19** | **1.000** | **39** | **1.000** | | |
| 20 | 0.967 | 40 | 0.974 | | |

$$\eta_3 = \sqrt{(1-\omega_x)(1-\omega_y)}$$

$$\omega_x = (\sum_{i=1}^{M} \frac{x_{io} - x_{iZ}}{x_{io}}) / M \ , \ \ \omega_y = (\sum_{j=1}^{R} \frac{y_{jZ} - y_{jo}}{y_{jo}}) / R \, .$$

If $(\omega_x + \omega_y)$ is positive, the efficiency score $\eta_2$ is less than 1. If it is negative, the efficiency score $\eta_2$ is larger than 1. Note that if DMU *o* is efficient in the additive model, either $(x_{io} - x_{iZ})$ is negative for some *i* or $(y_{jZ} - y_{jo})$ is negative for some *j*.

Figure 1 shows an example of reference points. If efficiency scores are derived by above procedures, the efficiency of DMU A is measured by the distance between DMU A and DMU E and efficiency of DMU B is measured by the distance between DMU A and DMU B. Therefore the efficiency score of DMU A may be smaller than DMU B. In this case the efficiency of DMU B must be measured by distances from DMU E. Therefore the following phase should be added to [3].

**<Phase 4>** Reconstruction of efficient facets based on DMUs which have $\eta_i \geq 1$ and derive the final efficiency scores based on distances from these facets.

## 3 Evaluation of Prefectures in Japan

We use input-output tables for the prefectures of Japan. We evaluate prefectures in terms of “import”, and “wages and salaries” as **inputs**, “private and government consumption”, and “export” as **outputs**, “sectoral outputs” as **the internal outputs**, and “sectoral inputs” as **the internal inputs**.

In SBM, efficiency scores are calculated based on $\eta_1$ , but, due to the shortcomings of ratio measure the efficiency scores shown in Table 1 were based on $\eta_2$.

Of the DMUs, 22 DMUs were efficient and 40 efficient facets were derived.

In (i) of phase 1, more than 22 DMUs were potentially efficient. Actually for DMU 31 there were 25 efficient DMUs.

In phase 3, twelve DMUs had efficiency scores greater or equal to than 1 and 11 efficient facets were derived. Table 2 shows efficiency scores $\eta_1$ and Table 3 shows DMU numbers on efficient facets in phase 3.

Final efficiency scores from phase 4 are shown in Table 4. Since the number of efficient DMUs is in phase 4 less than in phase1, the efficiency scores in phase 4 were less than those shown in Table 2. A comparison of SBM, phase 3 and the final efficiency scores is shown in Fig.2. The final efficiency scores of DMU 1 (Hokkaidou) and DMU 47 (Okinawa) were considerably smaller than those based on SBM. This difference is due to negative slacks, which give efficiency scores of 1 in SBM.

## 4 Minimum Distance Model in Network DEA

Network DEA programs deal with DMUs with internal sectors. Network DEA program take into consideration inputs and outputs between internal sectors, and measure the overall efficiency of the organizations, and measure the efficiency of each sectors [4]-[6]

Table 2 Efficiency scores in phase 3

| DMU | $\eta_1$ | $\eta_2$ | $\eta_3$ | DMU | $\eta_1$ | $\eta_2$ | $\eta_3$ | DMU | $\eta_1$ | $\eta_2$ | $\eta_3$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.867 | 0.926 | 0.925 | 21 | 0.941 | 0.970 | 0.970 | 41 | 0.989 | 0.994 | 0.994 |
| 2 | 0.959 | 0.979 | 0.978 | **22** | **1.021** | **1.010** | **1.010** | 42 | 0.994 | 0.997 | 0.997 |
| 3 | 0.939 | 0.968 | 0.967 | 23 | 0.988 | 0.994 | 0.994 | 43 | 0.980 | 0.990 | 0.990 |
| 4 | 0.999 | 0.999 | 0.999 | 24 | 0.992 | 0.996 | 0.996 | 44 | 0.953 | 0.976 | 0.976 |
| 5 | 0.986 | 0.993 | 0.993 | 25 | 0.978 | 0.989 | 0.989 | 45 | 0.952 | 0.975 | 0.974 |
| 6 | 0.919 | 0.956 | 0.956 | 26 | 0.985 | 0.993 | 0.993 | 46 | 0.978 | 0.989 | 0.989 |
| 7 | 0.977 | 0.989 | 0.989 | **27** | **1.063** | **1.031** | **1.031** | 47 | 0.894 | 0.942 | 0.941 |
| 8 | 0.982 | 0.991 | 0.991 | 28 | 0.974 | 0.987 | 0.987 | 48 | 0.971 | 0.985 | 0.985 |
| **9** | **1.002** | **1.001** | **1.001** | **29** | **1.018** | **1.009** | **1.009** | 49 | 0.981 | 0.991 | 0.991 |
| **10** | **1.000** | **1.000** | **1.000** | **30** | **1.042** | **1.021** | **1.021** | **50** | **1.005** | **1.001** | **0.880** |
| **11** | **1.011** | **1.006** | **1.005** | **31** | **1.067** | **1.037** | **1.027** | 51 | 0.987 | 0.994 | 0.994 |
| 12 | 0.991 | 0.995 | 0.995 | 32 | 0.942 | 0.969 | 0.969 | **52** | **1.007** | **1.004** | **1.004** |
| **13** | **1.023** | **1.010** | **1.006** | 33 | 0.980 | 0.990 | 0.990 | 53 | 0.967 | 0.983 | 0.983 |
| **14** | **1.012** | **1.006** | **1.006** | 34 | 0.947 | 0.973 | 0.973 | 54 | 0.962 | 0.981 | 0.981 |
| 15 | 0.957 | 0.977 | 0.977 | 35 | 0.992 | 0.996 | 0.996 | | | | |
| 16 | 0.952 | 0.975 | 0.974 | 36 | 0.973 | 0.987 | 0.986 | | | | |
| 17 | 0.948 | 0.973 | 0.972 | 37 | 0.985 | 0.993 | 0.993 | | | | |
| 18 | 0.992 | 0.996 | 0.996 | 38 | 0.976 | 0.988 | 0.988 | | | | |
| 19 | 0.988 | 0.994 | 0.994 | 39 | 0.886 | 0.935 | 0.931 | | | | |
| 20 | 0.951 | 0.976 | 0.975 | 40 | 0.981 | 0.991 | 0.990 | | | | |

Table 4 Final efficiency scores

| DMU | $\eta_1$ | $\eta_2$ | $\eta_3$ | DMU | $\eta_1$ | $\eta_2$ | $\eta_3$ | DMU | $\eta_1$ | $\eta_2$ | $\eta_3$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.746 | 0.830 | 0.813 | 21 | 0.935 | 0.965 | 0.965 | 41 | 0.981 | 0.990 | 0.990 |
| 2 | 0.933 | 0.967 | 0.966 | **22** | **1.021** | **1.010** | **1.010** | 42 | 0.963 | 0.981 | 0.980 |
| 3 | 0.918 | 0.959 | 0.958 | 23 | 0.988 | 0.994 | 0.994 | 43 | 0.944 | 0.973 | 0.970 |
| 4 | 0.985 | 0.993 | 0.992 | 24 | 0.985 | 0.992 | 0.992 | 44 | 0.953 | 0.976 | 0.976 |
| 5 | 0.924 | 0.959 | 0.958 | 25 | 0.950 | 0.973 | 0.972 | 45 | 0.869 | 0.924 | 0.921 |
| 6 | 0.917 | 0.955 | 0.954 | 26 | 0.985 | 0.993 | 0.993 | 46 | 0.956 | 0.979 | 0.977 |
| 7 | 0.974 | 0.987 | 0.987 | **27** | **1.063** | **1.031** | **1.031** | 47 | 0.665 | 0.750 | 0.709 |
| 8 | 0.977 | 0.988 | 0.988 | 28 | 0.974 | 0.987 | 0.987 | 48 | 0.938 | 0.968 | 0.968 |
| **9** | **1.002** | **1.001** | **1.001** | **29** | **1.018** | **1.009** | **1.009** | 49 | 0.965 | 0.983 | 0.982 |
| **10** | **1.000** | **1.000** | **1.000** | **30** | **1.042** | **1.021** | **1.021** | **50** | **1.005** | **1.001** | **0.880** |
| **11** | **1.011** | **1.006** | **1.005** | **31** | **1.067** | **1.037** | **1.027** | 51 | 0.972 | 0.986 | 0.986 |
| 12 | 0.981 | 0.991 | 0.991 | 32 | 0.937 | 0.969 | 0.968 | **52** | **1.007** | **1.004** | **1.004** |
| **13** | **1.023** | **1.010** | **1.006** | 33 | 0.978 | 0.989 | 0.989 | 53 | 0.961 | 0.980 | 0.980 |
| **14** | **1.012** | **1.006** | **1.006** | 34 | 0.947 | 0.973 | 0.973 | 54 | 0.909 | 0.950 | 0.949 |
| 15 | 0.948 | 0.974 | 0.974 | 35 | 0.992 | 0.996 | 0.996 | | | | |
| 16 | 0.927 | 0.961 | 0.961 | 36 | 0.966 | 0.983 | 0.983 | | | | |
| 17 | 0.939 | 0.969 | 0.969 | 37 | 0.985 | 0.993 | 0.993 | | | | |
| 18 | 0.963 | 0.981 | 0.981 | 38 | 0.971 | 0.985 | 0.985 | | | | |
| 19 | 0.939 | 0.966 | 0.964 | 39 | 0.817 | 0.890 | 0.885 | | | | |
| 20 | 0.941 | 0.970 | 0.970 | 40 | 0.951 | 0.974 | 0.974 | | | | |

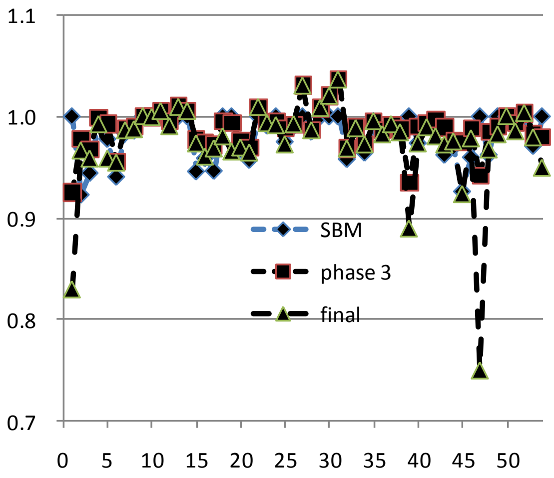


Fig. 2 Comparison of 2

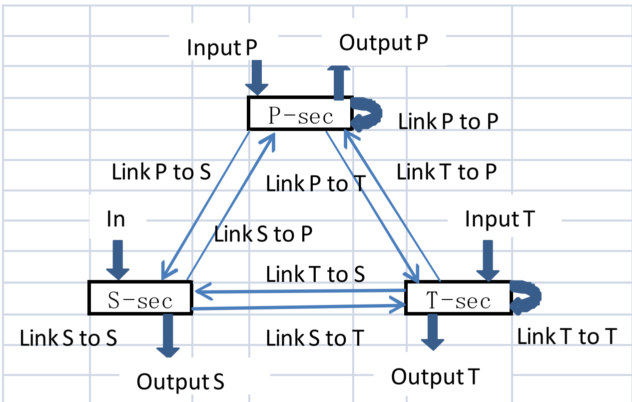


Fig. 3 Matrix type organization

Table 3 DMU numbers on efficient facets

| | | | |
|---|---|---|---|
| (9 | 10 | 22 | 30) |
| (11 | 14 | 27 | 30) |
| (11 | 27 | 29 | 30) |
| (13 | 14 | 22 | 27) |
| (13 | 14 | 22 | 50) |
| (13 | 14 | 27 | 50) |
| (13 | 22 | 27 | 30) |
| (13 | 27 | 30 | 31) |
| (13 | 27 | 50 | 52) |
| (14 | 22 | 27 | 30) |
| (14 | 27 | 50 | 52) |

[The first facet includes DMU 9, 10, 22 and 30.]

Table 5 Efficient scores of DMU 1 and 47

| DMU | SBM | phase 3 | final |
|---|---|---|---|
| 1 | **1.000** | 0.926 | 0.830 |
| 47 | **1.000** | 0.942 | 0.750 |

Conventional network DEA programs deal with organizations that have two special sectors: the starting sector and the ending sector. However, in organizations described in input-output tables, each sector produces goods for other sectors and at simultaneously receives goods from other sectors, as shown in Fig. 3. We refer to this type of organization as a matrix-type organization. In [7], we proposed algorithms for network DEA programs to measure the efficiency of matrix-type organizations.

However, [7] used maximum slacks in the same way as [6] did. In this section we discuss the minimum slacks.

In the free-link (FL) model of [7], the following formulation was proposed:

$$\rho = \min_{\lambda, s^-, s^+} \frac{\sum_{k=1}^{K} w_o^k \left(1 - \frac{1}{M^k} \sum_{m=1}^{M^k} s_m^{k-} / x_{mo}^k\right)}{\sum_{k=1}^{K} w_o^k \left(1 + \frac{1}{R^k} \sum_{r=1}^{R^k} s_r^{k+} / y_{ro}^k\right)}, \quad (1)$$

subject to

$$\boldsymbol{x}_o^k = \boldsymbol{x}^k \boldsymbol{\lambda}^k + \boldsymbol{s}^{k-}, \quad \boldsymbol{y}_o^k = \boldsymbol{y}^k \boldsymbol{\lambda}^k - \boldsymbol{s}^{k+}, \quad (2)$$

$$\mathbf{z}^{(k,h)} \boldsymbol{\lambda}^k = \mathbf{z}^{(k,h)} \boldsymbol{\lambda}^h \quad (\forall k \ \ \forall h), \quad (3)$$

$$\boldsymbol{\lambda} \geq 0, \boldsymbol{s}^- \geq 0, \boldsymbol{s}^+ \geq 0, \quad (4)$$

where $(\mathbf{x}_j^k, \mathbf{y}_j^k)$ is the inputs and outputs for sector *k* of DMU *j* and $\mathbf{z}_j^{(k,h)}$ is the quantity of link from sector *k* to sector *h*. In Eq. (1), 25 DMUs are efficient. Let the set of efficient DMUs be *EF*. Among the cases, *C*, where **"min"** in FL [7] was changed into **"max"** and the summation was over *EF*, there were some infeasible cases. Since not every point expressed by a linear combination of efficient DMUs is necessarily on the efficient frontier, we could not assume that the solutions of cases *C* solved the model, but infeasible solutions of cases *C* could be ruled out

In our procedures, the efficient facets must be derived. The efficient frontier includes not only inputs and outputs, but also links that have not yet been discussed. One possibly better idea is to derive efficient facets for each sector and the minimum slacks between their facets and DMU*o*. The analysis of the appropriateness of this idea is shown below.

When using $(\mathbf{x}_j^k, \mathbf{y}_j^k)$, efficient frontiers, $EF^{(k)}$, of sector *k* are constructed by following facets which are shown in numbers of efficient DMUs:

Sector 1: (6 39 46 ), (13 21 46 ),
(21 39 43), (21 39 46)

Sector 2: (1 12 30), (1 12 42)
(9 12 30) (9 30 35)
Sector 3: (12 13 47), (12 29 47),
(13 40 47), (29 39 47),
(31 39 43), (39 40 43),
(39 40 47)

Table 6 shows the efficiency scores obtained by ignoring the link constraints. If the existing link constraints, Eq. (3), are adopted, efficiency scores can not be obtained for 44 DMUs. Therefore, we propose a new network model as follows:

《New Formulation 》

[Step 1] We derive $D_f$ for the $a$-th facet $f_{ia}$ of efficient frontier in sector $i$.

$$D_{f_{abc}} = \min_{\lambda,s^-,s^+} \left[SI' + SO' + SL'\right], \ \boldsymbol{f}_{abc} = (f_{1a}, f_{2b}, f_{3c}), \tag{5}$$

$$SI' = \sum_{k=1}^{K}\{\frac{w_o^k}{M^k}\sum_{m=1}^{M^k}(s_{1m}^{k-} + s_{2m}^{k-}) / x_{mo}^k\},$$

Table 6 Efficiency scores without link constraints

| DMU | score | DMU | score | DMU | score |
|---|---|---|---|---|---|
| 1 | 0.983 | 21 | 0.952 | 41 | 0.952 |
| 2 | 0.964 | 22 | 0.950 | 42 | 0.995 |
| 3 | 0.944 | 23 | 0.958 | 43 | 0.994 |
| 4 | 0.975 | 24 | 0.943 | 44 | 0.952 |
| 5 | 0.980 | 25 | 0.888 | 45 | 0.964 |
| 6 | 0.949 | 26 | 0.966 | 46 | 0.969 |
| 7 | 0.968 | 27 | 0.966 | 47 | 0.918 |
| 8 | 0.973 | 28 | 0.971 | 48 | 0.954 |
| 9 | 0.981 | 29 | 0.984 | 49 | 0.966 |
| 10 | 0.980 | 30 | 0.988 | 50 | 0.982 |
| 11 | 0.988 | 31 | 0.958 | 51 | 0.957 |
| 12 | 0.997 | 32 | 0.929 | 52 | 0.961 |
| 13 | 0.982 | 33 | 0.970 | 53 | 0.958 |
| 14 | 0.989 | 34 | 0.948 | 54 | 0.972 |
| 15 | 0.942 | 35 | 0.981 | | |
| 16 | 0.934 | 36 | 0.966 | | |
| 17 | 0.950 | 37 | 0.959 | | |
| 18 | 0.844 | 38 | 0.972 | | |
| 19 | 0.966 | 39 | 0.987 | | |
| 20 | 0.959 | 40 | 0.986 | | |

$$SO' = \sum_{k=1}^{K}\{\frac{w_o^k}{R^k}\sum_{r=1}^{R^k}(s_{1r}^{k+} + s_{2r}^{k+}) / y_{ro}^k\},$$

$$SL' = \frac{1}{2K(K-1)}(SL_I' + SL_O'),$$

$$SL_I^{'} = \sum_{i \neq h} (s_1^{(i,h)-} + s_2^{(i,h)-}) / z_o^{(i,j)},$$

$$SL_O^{'} = \sum_{i \neq h} (s_1^{(i,h)+} + s_2^{(i,h)+}) / z_o^{(i,j)},$$

subject to

$$x_{om}^k = \sum_{j \in EF^{(k)}} x_{jm}^k \boldsymbol{\lambda}_j^k + s_{1m}^{k-} - s_{2m}^{k-} \ (m = 1,\ldots,M^k; \forall k), \tag{6}$$

$$y_{or}^k = \sum_{j \in EF^{(k)}} y_{jr}^k \lambda_j^k - (s_{1r}^{k+} - s_{2r}^{k+}) \ \ (r = 1,\ldots,R^k; \forall k), \tag{7}$$

$$\mathbf{z}_o^{(i,h)} = \sum_{j \in EF^{(i)}} \mathbf{z}_{\boldsymbol{j}}^{(i,h)} \boldsymbol{\lambda}_j^{(i)} - (\boldsymbol{s}_1^{(i,h)-} - \mathbf{s}_2^{(i,h)-}), \tag{8}$$

$$\mathbf{z}_o^{(i,h)} = \sum_{j \in EF^{(h)}} \mathbf{z}_{\boldsymbol{j}}^{(i,h)} \boldsymbol{\lambda}_j^{(h)} + (\boldsymbol{s}_1^{(i,h)+} - \mathbf{s}_2^{(i,h)+}), \qquad (\forall i, \forall h) \tag{9}$$

$$\lambda_j \geq 0, \ s_{1m}^{k\pm} \geq 0, \ s_{2r}^{k\pm} \geq 0 \quad (\forall k, m, r), \tag{10}$$

$$s_1^{(i,h)\pm} \geq 0, s_2^{(i,h)\pm} \geq 0 \ (\forall i, h). \tag{11}$$

[Step 2] We obtain the point $L^i$ which minimizes $D_{f_{abc}}$ in sector *i*.

[Step 3] We calculate efficiency scores, $\rho_2$ and $\rho_3$.

$$\rho_2 = 1 - (SI + SO + SL) / 3,$$

$$\rho_3 = \{(1 - SI)(1 - SO)(1 - SL)\}^{1/3},$$

$$(1 - SI)(1 - SO)(1 - SL) < 0 \Rightarrow \rho_3 = 0,$$

where

$$SI = \sum_{k=1}^{K} \{ \frac{w_o^k}{M^k} \sum_{m=1}^{M^k} (s_{1m}^{k-} - s_{2m}^{k-}) / x_{mo}^k \},$$

$$SO = \sum_{k=1}^{K} \{ \frac{w_o^k}{R^k} \sum_{r=1}^{R^k} (s_{1r}^{k+} - s_{2r}^{k+}) / y_{ro}^k \},$$

$$SL = \frac{1}{2K(K-1)} \sum_{i \neq h} (s_1^{(i,h)-} - s_2^{(i,h)-} + s_1^{(i,h)+} - s_2^{(i,h)+}) / z_o^{(i,j)},$$

$$\mathbf{x}_o^k - x_{L^k}^k = \mathbf{s}_1^{k-} - \mathbf{s}_2^{k-}, \quad y_{L^k}^k - \mathbf{y}_o^k = \mathbf{s}_1^{k+} - \mathbf{s}_2^{k+},$$

$$\mathbf{z}_{L^i}^{(i,h)} - \mathbf{z}_o^{(i,h)} = \mathbf{s}_1^{(i,h)-} - \mathbf{s}_2^{(i,h)-}, \ \mathbf{z}_o^{(i,h)} - \mathbf{z}_{L^h}^{(i,h)} = \mathbf{s}_1^{(i,h)+} - \mathbf{s}_2^{(i,h)+}.$$

The addition of *SL* and the modification of Eq. (3) to Eqs. (9) and (10) are new to this formulation. Efficiency scores in this model are shown in Table 7 and Fi.g.4. In DMU 14 (Kanagawa), $\rho_2$ and $\rho_3$ are not appropriate as efficiency scores because *SL* is large ( *SI* = 0.103, *SO* = −0.211 and *SL* = 0.618: see Table 8). $\Lambda$, the sum of the $\lambda$s, was introduced to reduce the effect of *SL* being large and the following efficiency scores are proposed:

$$\rho_4 = 1 - (SI + SO + SL/\Lambda)/3,$$

$$\rho_5 = \{(1 - SI)(1 - SO)(1 - SL/\Lambda)\}^{1/3},$$

$$\rho_6 = 1 - (SI + SO)/2.$$

Table 9 and Fig.5 shows efficiency scores $\rho_4$, $\rho_5$ and $\rho_6$. Efficiency scores of DMU 14 are larger than 1, where $\Lambda$=12.1595, *SI*= 0.103063, *SO*= –0.210753, and *SL* = 0.617901 so that $SL/\Lambda < |SO|$.

In DMU 37 (Kagawa), $\rho_4$ and $\rho_5$ were larger than 1, but $\rho_6$ was less than 1. This is because *SL* was large and negative. Similar results were obtained for DMU 5 (Akita), 12 (Chiba), 30 (Wakayama), 36 (Tokushima) and 43 (Kumamoto).

$\rho_4$ was nearly equal to $\rho_5$, but $\rho_6$ was different from these because $\rho_6$ does not take into account *SL*. Which score is the best for evaluation of efficiency is under study.

Table 7 Efficiency scores 2 and 3 in the new network model

| DMU | $\rho_2$ | $\rho_3$ | DMU | $\rho_2$ | $\rho_3$ | DMU | $\rho_2$ | $\rho_3$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 1.028 | 1.026 | 21 | 1.006 | 1.005 | 41 | 1.028 | 1.022 |
| 2 | 1.009 | 1.007 | 22 | 0.998 | 0.998 | 42 | 0.998 | 0.998 |
| 3 | 1.025 | 1.019 | 23 | 0.974 | 0.973 | 43 | 1.006 | 1.006 |
| 4 | 1.033 | 1.030 | 24 | 1.010 | 1.007 | 44 | 1.025 | 1.020 |
| 5 | 1.039 | 1.036 | 25 | 0.940 | 0.940 | 45 | 0.997 | 0.995 |
| 6 | 1.035 | 1.027 | 26 | 0.925 | 0.919 | 46 | 1.024 | 1.020 |
| 7 | 1.018 | 1.016 | 27 | 0.937 | 0.913 | 47 | 0.936 | 0.934 |
| 8 | 1.032 | 1.028 | 28 | 0.990 | 0.989 | 48 | 1.030 | 1.025 |
| 9 | 1.011 | 1.009 | 29 | 0.995 | 0.995 | 49 | 1.026 | 1.022 |
| 10 | 1.011 | 1.011 | 30 | 1.032 | 1.030 | 50 | 0.958 | 0.957 |
| 11 | 0.952 | 0.947 | 31 | 1.004 | 1.002 | 51 | 0.999 | 0.998 |
| 12 | 1.020 | 1.019 | 32 | 1.014 | 1.007 | 52 | 0.963 | 0.961 |
| 13 | 0.982 | 0.967 | 33 | 1.018 | 1.017 | 53 | 1.018 | 1.014 |
| 14 | 0.830 | 0.746 | 34 | 0.999 | 0.995 | 54 | 1.007 | 1.006 |
| 15 | 1.012 | 1.007 | 35 | 1.026 | 1.024 | | | |
| 16 | 0.982 | 0.980 | 36 | 1.028 | 1.025 | | | |
| 17 | 0.994 | 0.993 | 37 | 1.084 | 1.070 | | | |
| 18 | 0.947 | 0.945 | 38 | 1.026 | 1.023 | | | |
| 19 | 0.993 | 0.993 | 39 | 1.003 | 1.002 | | | |
| 20 | 1.021 | 1.018 | 40 | 1.019 | 1.016 | | | |

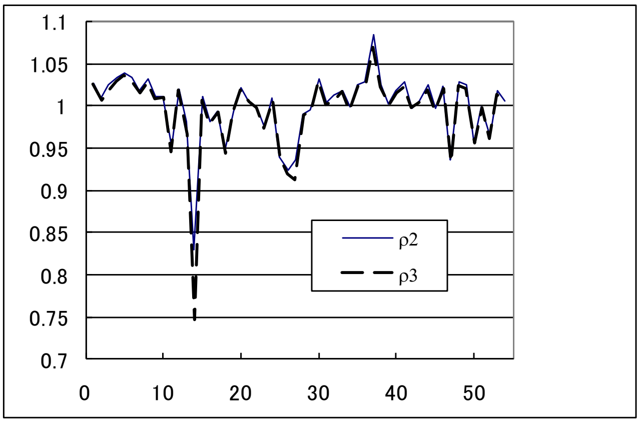


Fig. 4 Efficiency scores 2 and 3 in the new network model

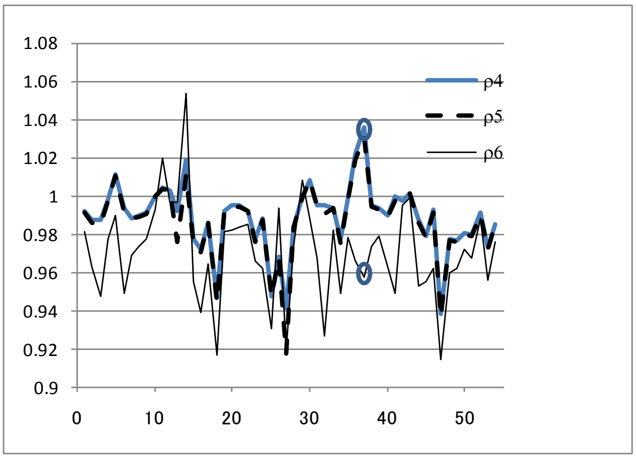


Fig. 5 Efficiency scores 4, 5 and 6 in the new network model

## 5 Conclusion

We applied the unified DEA model to evaluating efficiencies of prefectures and compared the efficiency scores with those obtained using SBM. Efficiency scores of two DMUs, 1 and 47, became considerably smaller than those obtained using SBM. We defined prefectures as matrix-type organizations and could evaluate their efficiencies, using a new network DEA model which minimizes slacks and relaxes link constraints between sectors.

### Appendix Alternative Formulation of Sec. 4

The following formulation is equivalent with Step 1 and 2 of section 4.

$$D_{f} = \min_{\lambda, s^-, s^+} \left[ SI + SO + SL \right] \quad f = f_1 \ f_2 \ f_3$$

$$SI' = \sum_{k=1}^{K} \{ \frac{w_o^k}{M^k} \sum_{m=1}^{M^k} (s_{1m}^{k-} + s_{2m}^{k-}) / x_{mo}^k \},$$

$$SO' = \sum_{k=1}^{K} \{ \frac{w_o^k}{R^k} \sum_{r=1}^{R^k} (s_{1r}^{k+} - s_{2r}^{k+}) / y_{ro}^k \},$$

$$SL' = \frac{1}{2K(K-1)} \sum_{i \neq h} (s_1{}^{(i,h)-} + s_2{}^{(i,h)-} + s_1{}^{(i,h)+} + s_2{}^{(i,h)+}) / z_o^{(i,j)},$$

$$\boldsymbol{x}_o^k = \sum_{j \in EF^{(k)}} \boldsymbol{x}_j^k \boldsymbol{\lambda}_j^{(k)} + \boldsymbol{s}_1^{k-} - \boldsymbol{s}_2^{k-},$$

$$\boldsymbol{y}_o^k = \sum_{j \in EF^{(k)}} \boldsymbol{y}_j^k \boldsymbol{\lambda}_j^{(k)} - (\boldsymbol{s}_1^{k+} - \boldsymbol{s}_2^{k+}) \quad (k = 1, 2, ... K),$$

$$\boldsymbol{z}_o^{(i,h)} = \sum_{j \in EF^{(i)}} \boldsymbol{z}_j^{(i,h)} \boldsymbol{\lambda}_j^{(i)} + \boldsymbol{s}_1^{(i,h)-} - \boldsymbol{s}_2^{(i,h)-},$$

$$\boldsymbol{z}_o^{(i,h)} = \sum_{j \in EF^{(h)}} \boldsymbol{z}_j^{(i,h)} \boldsymbol{\lambda}_j^{(h)} - (\boldsymbol{s}_1^{(i,h)+} - \boldsymbol{s}_2^{(i,h)+}),$$

$$s_1{}^{(i,h)\pm} \geq 0, s_2{}^{(i,h)\pm} \geq 0 \quad (\forall i \neq h),$$

$$\boldsymbol{\lambda} \geq \boldsymbol{0}, \boldsymbol{s}^- \geq \boldsymbol{0}, \boldsymbol{s}^+ \geq \boldsymbol{0},$$

$$-\sum_{i=1}^{m} v_i^{(k)} x_{ij}^k + \sum_{h=1}^{s} u_h^{(k)} y_{hj}^k + d_j^{(k)} = 0$$

$$(j \in EF^{(k)}; k = 1, 2, ..., K),$$

$$M(1 - b_j^{(k)}) \geq \lambda_j^{(k)} \geq 0, \quad M \cdot b_j^{(k)} \geq d_j^{(k)} \geq 0 \ (j \in EF^{(k)}),$$

$$v_i^{(k)} \geq 1 / x_{io}^k (i = 1, ..., m), u_h^{(k)} \geq 1 / y_{io}^k$$

$$(h = 1, ..., s); \quad (k = 1, ..., K),$$

$$b_j^{(k)} \in \{0, 1\} \quad (j \in EF^{(k)}; k = 1, ..., K).$$

However, this is not a simple linear program because of binary $b_j^{(k)}$ s. Therefore, we cannot necessarily obtain the same solutions with Sec. 4 for all DMUs.

## Acknowledges

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# References

[1] Tone T. A slacks-based measure of efficiency in data envelopment analysis, *European Journal of Operational Research*, 2001, 130(3): 498-509.

**[2]** Ueda T. and Amatatsu H. Evaluation of chemical companies using the ADD-MIN model, *Proceedings of DEA Symposium 2008*, 2008: 96-101.

[3] Ueda T. and Amatatsu H. Proposition of the unified DEA model including

super-efficiency measure, *Proceedings of DEA Symposium 2009*, 2009: 86-90.
[4] Bogetoft P., Fä re R. and Grosskopf S. Thoughts about Network DEA, *Proceedings of DEA Symposium 2008*, 2008: 1-4.
[5] Tone K. and Tsutsui M. Several Properties of the slacks-based network DEA models, *Proceedings of DEA Symposium 2008*, 2008: 64-69.
[6] Tone K. and Tsutsui M. Network DEA: A slacks-based measure approach. *European Journal of Operations Research*, 2009, 197(1): 243-252.
[7] Amatatsu H. and Ueda T. Input-Output Tables and Network DEA: Efficiencies of the 47 Prefectures of Japan. *Proceedings of DEA Symposium 2009*, 2009: 91-97.

Table 8 Sums of slacks, SI, SO and SL, in the new network model

| DMU | SI | SO | SL | DMU | SI | SO | SL | DMU | SI | SO | SL |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.0146 | 0.0227 | −0.1200 | 21 | 0.0648 | −0.0312 | −0.0521 | 41 | 0.0608 | 0.0419 | −0.1879 |
| 2 | 0.0823 | −0.0080 | −0.1027 | 22 | 0.0170 | 0.0125 | −0.0241 | 42 | 0.0181 | −0.0080 | −0.0041 |
| 3 | 0.0838 | 0.0220 | −0.1813 | 23 | 0.0931 | −0.0251 | 0.0086 | 43 | 0.0101 | −0.0085 | −0.0193 |
| 4 | 0.0411 | 0.0034 | −0.1449 | 24 | 0.0806 | −0.0052 | −0.1064 | 44 | 0.0838 | 0.0110 | −0.1703 |
| 5 | 0.0114 | 0.0097 | −0.1366 | 25 | 0.0909 | 0.0483 | 0.0412 | 45 | 0.0126 | 0.0772 | −0.0815 |
| 6 | 0.0080 | 0.0939 | −0.2062 | 26 | 0.0494 | −0.0360 | 0.2125 | 46 | 0.0279 | 0.0480 | −0.1482 |
| 7 | 0.0301 | 0.0326 | −0.1166 | 27 | 0.3318 | −0.1699 | 0.0274 | 47 | 0.0259 | 0.1451 | 0.0219 |
| 8 | 0.0165 | 0.0367 | −0.1482 | 28 | 0.0446 | 0.0090 | −0.0226 | 48 | 0.0778 | 0.0021 | −0.1688 |
| 9 | −0.0006 | 0.0466 | −0.0780 | 29 | 0.0248 | −0.0408 | 0.0302 | 49 | 0.0233 | 0.0532 | −0.1532 |
| 10 | 0.0198 | −0.0053 | −0.0486 | 30 | 0.0144 | 0.0090 | −0.1181 | 50 | 0.0679 | −0.0123 | 0.0717 |
| 11 | 0.0151 | −0.0542 | 0.1832 | 31 | 0.0617 | 0.0030 | −0.0764 | 51 | 0.0658 | −0.0010 | −0.0621 |
| 12 | 0.0086 | −0.0022 | −0.0653 | 32 | 0.0650 | 0.0815 | −0.1882 | 52 | 0.0780 | −0.0551 | 0.0870 |
| 13 | 0.2184 | −0.2119 | 0.0463 | 33 | 0.0383 | −0.0016 | −0.0921 | 53 | 0.0632 | 0.0250 | −0.1416 |
| **14** | **0.1031** | **−0.2108** | **0.6179** | 34 | 0.1125 | −0.0109 | −0.0972 | 54 | 0.0155 | 0.0323 | −0.0675 |
| 15 | 0.1213 | −0.0317 | −0.1255 | 35 | 0.0085 | 0.0358 | −0.1222 | | | | |
| 16 | 0.0875 | 0.0339 | −0.0666 | 36 | 0.0558 | 0.0121 | −0.1528 | | | | |
| 17 | 0.0404 | 0.0311 | −0.0530 | 37 | 0.0611 | 0.0236 | −0.3364 | | | | |
| 18 | 0.1502 | 0.0163 | −0.0079 | 38 | 0.0422 | 0.0108 | −0.1298 | | | | |
| 19 | 0.0272 | 0.0098 | −0.0169 | 39 | 0.0154 | 0.0263 | −0.0505 | | | | |
| 20 | −0.0351 | 0.0717 | −0.0995 | 40 | 0.0439 | 0.0301 | −0.1298 | | | | |

Table 9 Efficiency scores 4, 5 and 6 in the new network model

| DMU | $\rho_4$ | $\rho_5$ | $\rho_6$ | DMU | $\rho_4$ | $\rho_5$ | $\rho_6$ | DMU | $\rho_4$ | $\rho_5$ | $\rho_6$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.992 | 0.992 | 0.981 | 21 | 0.995 | 0.994 | 0.983 | 41 | 1.000 | 0.997 | 0.949 |
| 2 | 0.987 | 0.986 | 0.963 | 22 | 0.992 | 0.992 | 0.985 | 42 | 0.997 | 0.997 | 0.995 |
| 3 | 0.987 | 0.985 | 0.947 | 23 | 0.977 | 0.976 | 0.966 | 43 | 1.001 | 1.001 | 0.999 |
| 4 | 0.998 | 0.997 | 0.978 | 24 | 0.988 | 0.987 | 0.962 | 44 | 0.987 | 0.986 | 0.953 |
| 5 | 1.011 | 1.011 | 0.989 | 25 | 0.948 | 0.947 | 0.930 | 45 | 0.979 | 0.978 | 0.955 |
| 6 | 0.994 | 0.991 | 0.949 | 26 | 0.968 | 0.967 | 0.993 | 46 | 0.993 | 0.992 | 0.962 |
| 7 | 0.988 | 0.988 | 0.969 | 27 | 0.942 | 0.917 | 0.919 | 47 | 0.938 | 0.936 | 0.914 |
| 8 | 0.989 | 0.989 | 0.973 | 28 | 0.984 | 0.983 | 0.973 | 48 | 0.977 | 0.976 | 0.960 |
| 9 | 0.991 | 0.991 | 0.977 | 29 | 0.999 | 0.999 | 1.008 | 49 | 0.976 | 0.976 | 0.962 |
| 10 | 1.000 | 0.999 | 0.993 | 30 | 1.008 | 1.008 | 0.988 | 50 | 0.980 | 0.980 | 0.972 |
| 11 | 1.004 | 1.004 | 1.020 | 31 | 0.995 | 0.994 | 0.968 | 51 | 0.979 | 0.979 | 0.968 |
| 12 | 1.003 | 1.003 | 0.997 | 32 | 0.995 | 0.990 | 0.927 | 52 | 0.991 | 0.990 | 0.989 |
| 13 | 0.992 | 0.976 | 0.997 | 33 | 0.994 | 0.993 | 0.982 | 53 | 0.972 | 0.972 | 0.956 |
| 14 | 1.019 | 1.010 | 1.054 | 34 | 0.977 | 0.975 | 0.949 | 54 | 0.985 | 0.985 | 0.976 |
| 15 | 0.978 | 0.975 | 0.955 | 35 | 0.999 | 0.998 | 0.978 | | | | |
| 16 | 0.972 | 0.970 | 0.939 | 36 | 1.022 | 1.019 | 0.966 | | | | |
| 17 | 0.986 | 0.986 | 0.964 | 37 | 1.036 | 1.030 | 0.958 | | | | |
| 18 | 0.947 | 0.944 | 0.917 | 38 | 0.995 | 0.994 | 0.973 | | | | |
| 19 | 0.992 | 0.992 | 0.981 | 39 | 0.993 | 0.993 | 0.979 | | | | |
| 20 | 0.995 | 0.994 | 0.982 | 40 | 0.990 | 0.989 | 0.963 | | | | |