E-Convexity of the Optimal Value Function in Parametric Nonlinear Programming

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Abstract
Consider a general parametric optimization problem \( P(\varepsilon) \) of the form \( \min_{x} f(x, \varepsilon) \), s.t. \( x \in R(\varepsilon) \). Convexity and generalized convexity properties of the optimal value function \( f^* \) and the solution set map \( S^* \) form an important part of the theoretical basis for sensitivity, stability, and parametric analysis in mathematical optimization. Fiacco and Kyparisis [1] systematically discussed the convexity and concavity of \( f^* \) for the above parametric program \( P(\varepsilon) \) and its several special forms. In this paper, we extend these main results in [1] to the E-convexity of \( f^* \) by introducing E-convexity of set-valued maps.

Keywords
Optimal value function; E-convex functions; E-quasiconvex functions; E-convex set-valued maps

1 Introduction
Let \( \mathbb{R}^n \) denote the \( n \)-dimensional Euclidean space. We consider a general parametric optimization problem of the form

\[
P(\varepsilon) \begin{cases} 
\min_{x} f(x, \varepsilon) \\
\text{s.t. } x \in R(\varepsilon),
\end{cases}
\]

where \( f : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^1 \) and \( R \) is a set-valued map from \( \mathbb{R}^k \) to \( \mathbb{R}^n \), as well as several specializations of this problem. The optimal value function \( f^* \) of problem \( P(\varepsilon) \) (sometimes called the perturbation function or the marginal function) is defined as

\[
f^* = \begin{cases} 
\inf_{x} \{ f(x, \varepsilon) | x \in R(\varepsilon) \}, & \text{if } R(\varepsilon) \neq \emptyset, \\
+\infty, & \text{if } R(\varepsilon) = \emptyset.
\end{cases}
\]

and the solution set-valued mappings \( S^* \) is defined by

\[
S^*(\varepsilon) = \{ x \in R(\varepsilon) | f(x, \varepsilon) = f^*(\varepsilon) \}.
\]

We also consider the following several special programs of \( P(\varepsilon) \):

\[
P_i(\varepsilon) \begin{cases} 
\min_{x} f(x, \varepsilon) \\
\text{s.t. } g_i(x, \varepsilon) \leq 0, i = 1, 2, \ldots, m, \\
h_j(x, \varepsilon) = 0, j = 1, 2, \ldots, p,
\end{cases}
\]
where $S \subset \mathbb{R}^n, g_i : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^1, i = 1, 2, \ldots, m, h_j : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^1, j = 1, 2, \ldots, p,$
i.e., with $R$ defined by

$$R(\varepsilon) = \{x \in S | g_i(x, \varepsilon) \leq 0, i = 1, 2, \ldots, m, h_j(x, \varepsilon) = 0, j = 1, 2, \ldots, p\}.$$  

Similarly, generalized convexity properties of the optimal value function $f^*$ and the solution set-valued map $S^*$, such as continuity, differentiability, and so forth, form a theoretical basis for sensitivity, stability, and parametric analysis in nonlinear optimization. From the mid-1970s to the mid-1980s, the study of this area has been obtained intensively. Many papers had tried to unify these theories and methodologies, for instance [2-4]. Until 1986, Fiacco and Kyparisis[1] have systematically discussed the convexity and concavity of $f^*$ for the above parametric program $P(\varepsilon)$ and its several special forms. Similarly, generalized convexity properties of the optimal value function $f^*$ and the solution set map $S^*$, also play a role of theoretical basis for sensitivity, stability and parametric analysis in nonlinear programming. Zhang[5] discussed preinvexity and preincavity properties of $f^*$.

Recently, Youness [6] introduced a class of sets and a class of functions called $E$-convex sets and $E$-convex functions by relaxing the definitions of convex sets and convex functions, which has some important applications in various branches of mathematical sciences[7-9].

Motivated both by earlier research works and by the importance of the concepts of convexity and generalized convexity, we introduce the concepts of $E$-convex set-valued map and essentially $E$-convex set-valued map, and then develop some basic properties of $E$-convex and essentially $E$-convex set-valued maps. Based on these new concepts, $E$-convexity properties of the optimal value function $f^*$ for the parametric optimization problem $P(\varepsilon)$ and its several special forms are considered.

2 $E$-convexity of set-valued maps

In this section, we introduce two concepts of generalized convexity of set-valued maps. Throughout this section, $M$ is a nonempty subset in $\mathbb{R}^k$, and $R$ is a set-valued map from $M$ to $\mathbb{R}^n$.

**Definition 2.1**([6]) A set $M$ is said to be $E$-convex if there is a map $E : \mathbb{R}^k \to \mathbb{R}^k$ such that

$$(1 - \lambda)E(x) + \lambda E(y) \in M,$$

for each $x, y \in M$ and $\lambda \in [0, 1]$.
Lemma 2.1. ([6]) If a set $M$ is $E$-convex, then $E(M) \subset M$.

It is known from Lemma 2.1 that $E(M) \subseteq M$. Hence, for any set-valued map $R$, we have the following observations:

Observation (a) The set-valued map $R \circ E : M \to 2^{\mathbb{R}^n}$ defined by

$$(R \circ E)(x) = R(E(x))$$

for all $x \in M$ is well defined.

Observation (b) The Restriction $\tilde{R} : E(M) \to 2^{\mathbb{R}^n}$ of $R : M \to 2^{\mathbb{R}^n}$ to $E(M)$ defined by

$$\tilde{R}(\bar{x}) = R(\bar{x})$$

for all $\bar{x} \in E(M)$ is well defined.

Definition 2.2. ([1]) Let $M$ be a convex set.

(1) The set-valued map $R$ is called convex on $M$ if, for any $\varepsilon_1, \varepsilon_2 \in M$ and $\lambda \in [0, 1]$, we have

$$\lambda R(e_1) + (1 - \lambda)R(e_2) \subseteq R(\lambda e_1 + (1 - \lambda)e_2).$$

(2) The set-valued map $R$ is called essentially convex on $M$ if, for any $\varepsilon_1, \varepsilon_2 \in M$, $\varepsilon_1 \neq \varepsilon_2$ and $\lambda \in [0, 1]$, we have

$$\lambda R(e_1) + (1 - \lambda)R(e_2) \subseteq R(\lambda e_1 + (1 - \lambda)e_2).$$

Based on the concept of convex set-valued maps and essentially convex set-valued maps, we introduce the concepts of $E$-convex set-valued maps and essentially $E$-convex set-valued maps.

Definition 2.3. (1) The set-valued map $R$ is called $E$-convex on $M$ if there is a map $E : \mathbb{R}^k \to \mathbb{R}^k$ such that $M$ is an $E$-convex set and

$$\lambda (R \circ E)(e_1) + (1 - \lambda)(R \circ E)(e_2) \subseteq R(\lambda E(e_1) + (1 - \lambda)E(e_2)),$$

for any $e_1, e_2 \in M$ and $\lambda \in [0, 1]$.

(2) The set-valued map $R$ is called essentially $E$-convex on $M$ if there is a map $E : \mathbb{R}^k \to \mathbb{R}^k$ such that $M$ is an $E$-convex set and

$$\lambda (R \circ E)(e_1) + (1 - \lambda)(R \circ E)(e_2) \subseteq R(\lambda E(e_1) + (1 - \lambda)E(e_2)),$$

for any $e_1, e_2 \in M$, $E(e_1) \neq E(e_2)$ and $\lambda \in [0, 1]$.

Remark 2.1. If $R$ is convex (resp. essentially convex) on $M$, then $R$ is $E$-convex (resp. essentially $E$-convex) on $M$.

Remark 2.2. If $R$ is $E$-convex on $M$, then it is essentially $E$-convex on $M$. However, the converse is not true. See example 2.1.

Remark 2.3. If $R$ is $E$-convex on $M$, then it is convex-valued with respect to $E$ on $M$, i.e., $(R \circ E)(e)$ at each $e \in M$ is a convex set. However, An essentially convex set-valued map may not be convex-valued with respect to $E$ at the boundary points of $M$, as shown below.
Example 2.1. Let $E : \mathbb{R}^2 \to \mathbb{R}^2$ be an identify map, $R : \mathbb{R}^2 \to \mathbb{R}^1$ defined by

$$R(\varepsilon_1, \varepsilon_2) = \begin{cases} 
[0, 1], & \text{if } \varepsilon_1^2 + \varepsilon_2^2 < 1, \\
\{0\} \cup \{1\}, & \text{if } \varepsilon_1^2 + \varepsilon_2^2 = 1, \\
0, & \text{if } \varepsilon_1^2 + \varepsilon_2^2 > 1.
\end{cases}$$

and

$$M = \{(\varepsilon_1, \varepsilon_2) | \varepsilon_1^2 + \varepsilon_2^2 \leq 1\}.$$ 

It is easy to check that $R$ is essentially $E$-convex on $M$, but $(R \circ E)(\varepsilon_1, \varepsilon_2)$ is not convex if $\varepsilon_1^2 + \varepsilon_2^2 = 1$.

From now on, let $E$ be a map from $\mathbb{R}^k$ to $\mathbb{R}^k$ and $M$ be a nonempty $E$-convex set.

**Proposition 2.1.** Let $R$ be $E$-convex (resp. essentially $E$-convex) on $M$. Then the restriction, say $\bar{R} : C \to 2^{\mathbb{R}^n}$, of $R$ to any nonempty convex subset $C$ of $E(M)$ is convex (resp. essentially convex) on $C$.

**Proof.** Let $C \subset E(M)$ be convex, and let $\bar{\varepsilon}_1, \bar{\varepsilon}_2 \in C$ ($\bar{\varepsilon}_1$ and $\bar{\varepsilon}_2$ may not be distinct). Then there exist $\varepsilon_1, \varepsilon_2 \in M$ such that $\varepsilon_1 = E(\varepsilon_1)$ and $\varepsilon_2 = E(\varepsilon_2)$. Since $E$ is convex on $M$, it follows from the $E$-convexity of $R$ that

$$\lambda \bar{R}(\bar{\varepsilon}_1) + (1 - \lambda) \bar{R}(\bar{\varepsilon}_2) = \lambda \bar{R}(E(\varepsilon_1)) + (1 - \lambda) \bar{R}(E(\varepsilon_2)) = \lambda (R \circ E)(\varepsilon_1) + (1 - \lambda)(R \circ E)(\varepsilon_2) \subset R(\lambda E(\varepsilon_1) + (1 - \lambda)E(\varepsilon_2))$$

for all $\lambda \in [0, 1]$. Hence, $\bar{R}$ is convex on $C$.

**Corollary 2.1.** Let $R$ be $E$-convex (resp. essentially $E$-convex) on $M$. If $E(M) \subset M$ is a convex set, then the restriction $\bar{R} : E(M) \to 2^{\mathbb{R}^n}$ of $R$ to $E(M)$ is convex (resp. essentially convex) on $E(M)$.

**Proposition 2.2.** Let $E(M) \subset M$ be a convex set. If the restriction $\bar{R} : E(M) \to 2^{\mathbb{R}^n}$ of $R$ to $E(M)$ is convex (resp. essentially convex) on $E(M)$, then $R$ is $E$-convex (resp. essentially $E$-convex) on $M$.

**Proof.** Let $\varepsilon_1, \varepsilon_2 \in M$. Then $E(\varepsilon_1), E(\varepsilon_2) \in E(M)$, and by the convexity of $E(M)$, we can obtain $\lambda E(\varepsilon_1) + (1 - \lambda)E(\varepsilon_2) \in E(M)$ for all $\lambda \in [0, 1]$. Since $\bar{R}$ is convex on $E(M)$, we have

$$\lambda (R \circ E)(\varepsilon_1) + (1 - \lambda)(R \circ E)(\varepsilon_2) = \lambda \bar{R}(E(\varepsilon_1)) + (1 - \lambda)\bar{R}(E(\varepsilon_2)) = R(\lambda E(\varepsilon_1) + (1 - \lambda)E(\varepsilon_2)),$$

which shows $R$ is $E$-convex on $M$.

**Corollary 2.2.** Suppose that $E(M)$ be convex. Then $R$ is $E$-convex (resp. essentially $E$-convex) on $M$ if and only if its restriction $\bar{R} : E(M) \to 2^{\mathbb{R}^n}$ is convex (resp. essentially convex) on $E(M)$.

Let the map $I \times E : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n \times \mathbb{R}^k$ be

$$(I \times E)(x, \varepsilon) = (x, E(\varepsilon)), \quad \text{for any } (x, \varepsilon) \in \mathbb{R}^n \times \mathbb{R}^k.$$
Denote
\[ G(R) = \{(x, \varepsilon) | x \in R(\varepsilon), \varepsilon \in M\}. \]

It is easy to show that \( G(R) \) is \( I \times E \)-convex, if and only if
\[ (\lambda x_1 + (1 - \lambda)x_2, \lambda E(\varepsilon_1) + (1 - \lambda)E(\varepsilon_2)) \in G(R) \]
for each \((x_1, \varepsilon_1), (x_2, \varepsilon_2) \in G(R)\) and \(\lambda \in [0, 1]\).

**Proposition 2.3.** Suppose \( R \) is \( E \)-convex on \( M \). If \( R(\varepsilon) \subset (R \circ E)(\varepsilon) \) for each \( \varepsilon \in M \), then \( G(R) \) is \( I \times E \)-convex.

**Proof.** Let \((x_1, \varepsilon_1), (x_2, \varepsilon_2) \in G(R)\) and \(\lambda \in [0, 1]\). Then, \(x_1 \in R(\varepsilon_1), x_2 \in \varepsilon R(\varepsilon_2)\). By the assumption that \( R(\varepsilon) \subset (R \circ E)(\varepsilon) \), we obtain
\[ x_1 \in (R \circ E)(\varepsilon_1), \quad x_2 \in (R \circ E)(\varepsilon_2). \quad (2.1) \]

Since \( R \) is \( E \)-convex on \( M \) and (2.1), we get
\[ \lambda x_1 + (1 - \lambda)x_2 \in R(\lambda E(\varepsilon_1) + (1 - \lambda)E(\varepsilon_2)), \quad (2.2) \]
which means that \((\lambda x_1 + (1 - \lambda)x_2, \lambda E(\varepsilon_1) + (1 - \lambda)E(\varepsilon_2)) \in G(R)\). Therefore, \( G(R) \) is \( I \times E \)-convex.

**Proposition 2.4.** Suppose \( G(R) \) is \( I \times E \)-convex. If \((R \circ E)(\varepsilon) \subset R(\varepsilon)\) for each \( \varepsilon \in M \), then \( R \) is \( E \)-convex on \( M \).

**Proof.** Let \( \varepsilon_1, \varepsilon_2 \in M \) and \(\lambda \in [0, 1]\). Take arbitrary points \(x_1 \in (R \circ E)(\varepsilon_1), x_2 \in (R \circ E)(\varepsilon_2)\). Then, it follows from \((R \circ E)(\varepsilon) \subset R(\varepsilon)\) for each \( \varepsilon \in M \)
\[ x_1 \in R(\varepsilon_1), \quad x_2 \in R(\varepsilon_2). \quad (2.3) \]
That is,
\[ (x_1, \varepsilon_1), (x_2, \varepsilon_2) \in G(R). \quad (2.4) \]
Since \( G(R) \) is \( I \times E \)-convex and (2.4), we get
\[ (\lambda x_1 + (1 - \lambda)x_2, \lambda E(\varepsilon_1) + (1 - \lambda)E(\varepsilon_2)) \in G(R). \quad (2.5) \]
That is,
\[ \lambda x_1 + (1 - \lambda)x_2 \in R(\lambda E(\varepsilon_1) + (1 - \lambda)E(\varepsilon_2)), \]
which shows that \( R \) is \( E \)-convex on \( M \).

### 3 E-convexity of the optimal value function

In this section, we give the main results.

**Definition 3.1.** A function \( g : \mathbb{R}^k \to \mathbb{R}^1 \) is said to be \( E \)-convex on a set \( M \subset \mathbb{R}^k \)
if there is a map \( E : \mathbb{R}^k \to \mathbb{R}^k \) such that \( M \) is an \( E \)-convex set and
\[ g(\lambda E(\varepsilon_1) + (1 - \lambda)E(\varepsilon_2)) \leq \lambda g(E(\varepsilon_1)) + (1 - \lambda)g(E(\varepsilon_2)), \]
for each \( \varepsilon_1, \varepsilon_2 \in M \) and \(\lambda \in [0, 1]\).
It is easy to show that \( f : \mathbb{R}^n \times \mathbb{R}^k \rightarrow \mathbb{R}^1 \) is \((I \times E)\)-convex on \( \mathbb{R}^n \times M \), if and only if
\[
f(\lambda x_1 + (1 - \lambda)x_2, \lambda E(e_1) + (1 - \lambda)E(e_2)) \leq \lambda f(x_1, E(e_1)) + (1 - \lambda)f(x_2, E(e_2))
\]
for each \((x_1, e_1), (x_2, e_2) \in \mathbb{R}^n \times M \) and \( \lambda \in [0, 1] \).

**Theorem 3.1.** Consider the general parametric optimization problem \( P(\epsilon) \). If \( f \) is \((I \times E)\)-convex on the set \( \{(x, \epsilon)| x \in R(E(\epsilon)), \epsilon \in M\} \), \( R \) is essentially \( E \)-convex on \( M \), and \( M \) is \( E \)-convex, then \( f^* \) is \( E \)-convex on \( M \).

**Proof.** Let \( e_1, e_2 \in M, e_1 \neq e_2 \), and \( \lambda \in [0, 1] \). Then, by the \((I \times E)\)-convexity of \( f \) and essential \( E \)-convexity of \( R \), we obtain
\[
f^*(\lambda E(e_1) + (1 - \lambda)E(e_2))
= \inf_{x \in R(\lambda E(e_1) + (1 - \lambda)E(e_2))} f(x, \lambda E(e_1) + (1 - \lambda)E(e_2))
\leq \inf_{x_1 \in (R\epsilon(e_1)), x_2 \in (R\epsilon(e_2))} f(\lambda x_1 + (1 - \lambda) x_2, \lambda E(e_1) + (1 - \lambda) E(e_2))
= \lambda \inf_{x_1 \in (R\epsilon(e_1))} f(x_1, E(e_1)) + (1 - \lambda) \inf_{x_2 \in (R\epsilon(e_2))} f(x_2, E(e_2))
= \lambda f^*(E(e_1)) + (1 - \lambda)f^*(E(e_2)),
\]
i.e., \( f^* \) is \( E \)-convex on \( M \).

**Definition 3.2.**\(^{(10)}\) A function \( g : \mathbb{R}^k \rightarrow \mathbb{R}^1 \) is said to be \( E \)-quasiconvex on a set \( M \subset \mathbb{R}^k \) if there is a map \( E : \mathbb{R}^k \rightarrow \mathbb{R}^k \) such that \( M \) is an \( E \)-convex set and
\[
g(\lambda E(e_1) + (1 - \lambda)E(e_2)) \leq \max\{g(E(e_1)), g(E(e_2))\},
\]
for each \( e_1, e_2 \in M \) and \( \lambda \in [0, 1] \).

The functions \( g \) is said to be \( E \)-quasiconcave, if \(-g \) is \( E \)-quasiconvex; \( g \) is said to be \( E \)-quasiconvexmonotonic, if \( g \) both is \( E \)-quasiconvex and \( E \)-quasiconcave.

**Theorem 3.2.** Consider the parametric problem \( P_1(\epsilon) \). If \( g_i \) are \((I \times E)\)-quasiconcave on \( S \times M \), \( h_j \) are \((I \times E)\)-quasiconvexmonotonic on \( S \times M \), \( S \) is convex and \( M \) is \( E \)-convex, then \( R \), given by
\[
R(\epsilon) = \{x \in S| g_i(x, \epsilon) \leq 0, i = 1, 2, \cdots, m, h_j(x, \epsilon) = 0, j = 1, 2, \cdots, p\},
\]
is \( E \)-convex on \( M \).

**Proof.** Let \( e_1, e_2 \in M \) and take arbitrary points \( x_1 \in (R \circ E)(e_1), x_2 \in (R \circ E)(e_2) \). Then, \( x_1, x_2 \in S \),
\[
g_i(x_1, E(e_1)) \leq 0, g_i(x_2, (E(e_2)) \leq 0, i = 1, 2, \cdots, m
\]
and
\[
h_j(x_1, E(e_1)) = 0, h_j(x_2, (E(e_2)) = 0, j = 1, 2, \cdots, p.
\]
Since \( S \) is convex and \( M \) is \( E \)-convex, we have
\[
\lambda x_1 + (1 - \lambda)x_2 \in S \text{ and } \lambda E(e_1) + (1 - \lambda)E(e_2) \in M \quad \text{for any } \lambda \in [0, 1].
\]
By \((I \times E)\)-quasiconvexity of \(g_i\) on \(S \times M\) and (3.1), we obtain
\[
g_i(\lambda x_1 + (1 - \lambda)x_2, \lambda E(\varepsilon_1) + (1 - \lambda)E(\varepsilon_2)) \leq \max\{g_i(x_1, E(\varepsilon_1)), g_i(x_2, E(\varepsilon_2))\} \leq 0.
\]
Similarly, by \((I \times E)\)-quasimonotonic of \(h_j\) on \(S \times M\) and (3.2), we can get
\[
h_j(\lambda x_1 + (1 - \lambda)x_2, \lambda E(\varepsilon_1) + (1 - \lambda)E(\varepsilon_2)) = 0.
\]
Therefore, by (3.3-3.5), we obtain
\[
\lambda x_1 + (1 - \lambda)x_2 \in R(\lambda E(\varepsilon_1) + (1 - \lambda)E(\varepsilon_2)),
\]
which means that \(\lambda (R \circ E)(\varepsilon_1) + (1 - \lambda)(R \circ E)(\varepsilon_2) \subset R(\lambda E(\varepsilon_1) + (1 - \lambda)E(\varepsilon_2))\), i.e., \(R\) is \(E\)-convex on \(M\).

The following result is now immediate.

**Corollary 3.1.** Consider the parametric problem \(P_1(\varepsilon)\). if \(f\) is \((I \times E)\)-convex on the set \(\{(x, \varepsilon)| x \in R(E(\varepsilon)), \varepsilon \in M\}\), \(g_i\) are \((I \times E)\)-quasiconvex on \(S \times M\), \(h_j\) are \((I \times E)\)-quasimonotonic on \(S \times M\), \(S\) is convex and \(M\) is \(E\)-convex, then \(f^*\) is \(E\)-convex on \(M\).

**Proof.** This follows directly from Theorems 3.1 and Theorems 3.2.

The next result follows directly from Theorems 3.2.

**Theorem 3.3.** Consider the parametric problem \(P_2(\varepsilon)\). if \(g_i\) are \((I \times E)\)-quasiconvex on \(S \times M\), \(h_j\) are \((I \times E)\)-quasimonotonic on \(S \times M\), \(S\) is convex and \(M\) is \(E\)-convex, then \(R\), given by
\[
R(\varepsilon) = \{x \in S|g_i(x) \leq \varepsilon_i, i = 1, 2, \cdots, m, h_j(x) = \varepsilon_{m+j}, j = 1, 2, \cdots, p\},
\]
is \(E\)-convex on \(M\).

**Corollary 3.2.** Consider the parametric problem \(P_2(\varepsilon)\). if \(f\) is \((I \times E)\)-convex on the set \(\{(x, \varepsilon)| x \in R(E(\varepsilon)), \varepsilon \in M\}\), \(g_i\) are \((I \times E)\)-quasiconvex on \(S \times M\), \(h_j\) are \((I \times E)\)-quasimonotonic on \(S \times M\), \(S\) is convex and \(M\) is \(E\)-convex, then \(f^*\) is \(E\)-convex on \(M\).

**Proof.** This follows directly from Theorems 3.1 and Theorems 3.3.

**References**


