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Exponential Stability Analysis for Impulsive Neural Networks with Time-varying Delays

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Abstract

The main purpose of this paper is to further investigate the stability problem of impulsive neural networks with time-varying delays in the case that the underlying continuous delayed neural networks are unstable. By establishing an impulsive delayed differential inequality, some novel and less conservative criteria for global exponential stability of the equilibrium point of such model are derived analytically. It is shown that under certain conditions, impulses can make the underlying continuous unstable delayed neural networks globally exponentially stable. Our results have improved and generalized some published results and are help to design stability of neural networks when both delay effect and impulsive effect are taken into consideration. An example is also given to show the effectiveness of our results.

Keywords Global exponential stability; Impulsive; Neural networks; Time-varying delays

1 Introduction

In recent years, a lot of attention has been devoted to the research on Hopfield neural networks (HNNs) due to the fact that HNNs can be applied in a broad range of areas such as associative memory, repetitive learning, classification of patterns, and optimization problems. Meanwhile, it has been shown that these successful applications are greatly dependent on the dynamic behaviors of the neural networks, such as the uniqueness and asymptotic stability of equilibrium point of a designed neural networks. Therefore, the problem of stability analysis of HNNs has become a rather significant topic in both theoretical research and practical applications. On the other hand, due to the finite switching speed of the amplifiers and communication time, time delays is inevitably encountered in many neural networks, which may be a source of oscillation and instability both in biological and artificial neural networks. Thus, a great deal of research interests have been attracted to the stability analysis problem of HNNs with time delays. Based on different assumptions and by using different approaches, a great deal of sufficient conditions have been proposed to guarantee the asymptotic or exponential stability

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of HNNs with various types of time delays, such as constant, time-varying, or distributed (see[1, 2, 3, 4, 5, 6, 7, 8, 9, 10] and the references therein).

However, besides delay effect, impulsive phenomena can also be found in a wide variety of evolutionary process, particularly some biological systems such as biological neural networks and bursting rhythm models in pathology, in which many sudden and sharp changes occur instantaneously, in the form of impulse [11]. Therefore, neural network model with delay and impulsive effects should be more accurate to describe the evolutionary process of the systems. Consequently, in recent years there has been a growing interest in the stability analysis of impulsive neural networks with time delays since impulses can also affect the dynamical behaviors of the systems just as time delays. Some results related to this issues have been reported (see [11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21] and the reference therein). However, most of the aforementioned results have only focused on the case that the underlying continuous system without impulsive effects are stable. To the best of our knowledge, when the underlying continuous system are unstable, few results have been developed to utilize impulsive effects to make the underlying unstable system stable. In [20], by using the Lyapunov functions method and analysis technique, the authors obtained a result for the uniform stability of the equilibrium point of impulsive Hopfield-type neural networks systems with time delays. However, the result in [20] only refer to uniform stability, and the delays considered are constants. In [21], Liu and Wang established several exponential stability criteria for impulsive systems with time delay by employing the method of Lyapunov functionals, and it was shown that an unstable system can be made exponentially stable by an appropriate sequence of impulses. However, the results in [21] require that the length of the impulsive interval must be greater than the size of time-delay, which is too restrictive and conservative and there still exists open room for further improvement.

Motivated by the above discussions, the main purpose of this paper is to further investigate the stability problem of impulsive neural networks with time-varying delays in the case that the underlying continuous delayed neural networks are unstable. By establishing an impulsive delayed differential inequality, some novel and less conservative criteria for global exponential stability of the equilibrium point of such model are derived analytically. It is shown that under certain conditions, impulses can make the underlying continuous unstable delayed neural network globally exponentially stable. Our results have improved and generalized some published results and are help to design stability of neural networks when both delay effect and impulsive effect are taken into consideration. The effectiveness of our results are further illustrated by numerical example.

This paper is organized as follows. In Section 2, we introduce some basic definitions, notations and lemma. In Section 3, we obtain some criteria for the global exponential stability of the equilibrium point of the impulsive neural networks with time-varying delays. An example is given to illustrate the effectiveness of our theoretical results in section 4. Finally, concluding remarks are given in Section 5.

2 Preliminaries

Let R denote the set of real numbers, R^+ denote the set of nonnegative real numbers, Z^+ denote the set of positive integers and R^n denote the n-dimensional real space equipped with the Euclidean norm $\|\cdot\|$.

Consider the following impulsive Hopfield-type neural networks systems with time-varying delays:

$$\begin{cases} \dot{x}_i(t) = -c_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} g_j(x_j(t-\tau(t))) + u_i,\ t \neq t_k, t \geq t_0, \\ \Delta x_i(t)|_{t=t_k} = x_i(t_k^+) - x_i(t_k^-), \quad t = t_k, k \in Z^+, i = 1, \cdots, n, \end{cases} \tag{1}$$

where n denotes the number of neurons in the neural networks, $C = \mathrm{diag}(c_1, \ldots, c_n)$ is a diagonal matrix with $c_i > 0, i = 1, 2, \ldots, n$, x_i corresponds to the membrane potential of the unit i at time t, $\tau(t)$ is transmission delay, and satisfies $0 \leq \tau(t) \leq \gamma(\gamma$ is a constant); $f_j(x_j(t))$ and $g_j(x(t-\tau(t)))$ denote, respectively, the measures of response or activation to its incoming potentials of the unit j at time t and $t-\tau(t)$; u_i is the constant input from the outside of the network; $A = (a_{ij})_{n\times n}$ indicates the strength of the neuron interconnections within the network at time t; $B = (b_{ij})_{n\times n}$ indicates the strength of the neuron interconnections within the network at time $t-\tau(t)$; the fixed moments t_k satisfy $0 \leq t_0 < t_1 < t_2 < \cdots < t_k < \cdots$, and $\lim_{k\to\infty} t_k = +\infty$, $\Delta x_i(t)|_{t=t_k} = x_i(t_k^+) - x_i(t_k^-)$ denotes the jumps in the state variable at the time instants t_k, where $x_i(t^+) = \lim_{s\to t^+} x_i(s)$, $x_i(t^-) = \lim_{s\to t^-} x_i(s)$. Without loss of generality, we assume that $\lim_{t\to t_k^+} x_i(t) = x_i(t_k)$, which means $x_i(t)$ is continuous from the right. The initial conditions of Eq. (1) are given by $x_i(t) = \phi_i(t) \in PC([t_0-\gamma, t_0], R^n)$, where $PC([t_0-\gamma, t_0], R^n)$ denotes the set of all functions of bounded variation and right-continuous on any compact subinterval of $[t_0-\gamma, t_0]$. For $\phi \in PC([t_0-\gamma, t_0], R^n)$, the norm of ϕ is defined by $\|\phi\|_\gamma = \sup_{t_0-\gamma\leq s\leq t_0} \|\phi(s)\|$.

In this paper, we assume that some conditions are satisfied so that the equilibrium point of system (1) does exist. Let $x^*(t) = (x_1^*, x_2^*, \cdots, x_n^*)^\top$ be an equilibrium point of the system (1). Impulsive operator is viewed as perturbation of the equilibrium point x^* of such system without impulsive effects. Moreover, we assume that $\Delta x_i(t)|_{t=t_k} = x_i(t_k^+) - x_i(t_k^-) = d_{ik}(x_i(t_k^-) - x_i^*), d_{ik} \in R, i = 1, \ldots, n$, and $k \in Z^+$. In order to simplify the equation, we make the following transformation: $y_i = x_i - x_i^*$, $i = 1, 2, \ldots, n$, then we obtain the following system:

$$\begin{cases} \dot{y}_i(t) = -c_i y_i(t) + \sum_{j=1}^{n} a_{ij} \tilde{f}_j(y_j(t)) + \sum_{j=1}^{n} b_{ij} \tilde{g}_j(y_j(t-\tau(t))),\ t \neq t_k, t \geq t_0, \\ y_i(t_k) = x_i(t_k) - x_i^*(t_k) = (1+d_{ik}) y_i(t_k^-), \quad t = t_k, k \in Z^+, i = 1, \cdots, n, \end{cases} \tag{2}$$

where $\tilde{f}_j(y_j(t)) = f_j(x_j^* + y_j(t)) - f_j(x_j^*)$, and $\tilde{g}_j(y_j(t-\tau(t))) = g_j(x_j^* + y_j(t-\tau(t))) - g_j(x_j^*)$. Clearly, in order to prove the global exponential stability of the equilibrium point of system (1), we just need to prove the global exponential stability of the trivial solution of the system (2).

Let $y(t) = (y_1(t), y_2(t), \ldots, y_n(t))^\top$, $D_k = \mathrm{diag}(1+d_{1k}, 1+d_{2k}, \ldots, 1+d_{nk})$, $\tilde{F}(y(t)) = (\tilde{f}_1(y_1(t)), \tilde{f}_2(y_2(t)), \ldots, \tilde{f}_n(y_n(t)))^\top$, $\tilde{G}(y_\tau(t)) = (\tilde{g}_1(y_1(t-\tau(t))), \tilde{g}_2(y_2(t-\tau(t))), \ldots, \tilde{g}_n(y_n(t-\tau(t))))^\top$, then the system (2) can be rewritten as

$$\begin{cases} \dot{y}(t) = -Cy(t) + A\tilde{F}(y(t)) + B\tilde{G}(y_\tau(t)),\ t \neq t_k, t \geq t_0, \\ y(t_k) = D_k y(t_k^-), \qquad t = t_k, k \in Z^+, \end{cases} \tag{3}$$

Throughout this paper, we assume that the activation function f_r, and g_r are continuous, and satisfy assumption (A_1) or $(\tilde{A}_1)$ [22, 23]:

(A_1) There exist the constants $k_r, l_r > 0$, $r = 1,2,\ldots,n$, such that

$$0 \leq \frac{f_r(x_1) - f_r(x_2)}{x_1 - x_2} \leq k_r,$$
$$|g_r(x_1) - g_r(x_2)| \leq l_r|x_1 - x_2|, \quad r = 1,2,\ldots,n,$$

for any two different $x_1, x_2 \in R$.

$(\tilde{A}_1)$ There exist the constants $k_r, l_r > 0$, $r = 1,2,\ldots,n$, such that

$$|f_r(x_1) - f_r(x_2)| \leq k_r|x_1 - x_2|,$$
$$|g_r(x_1) - g_r(x_2)| \leq l_r|x_1 - x_2|, \quad r = 1,2,\ldots,n,$$

for any two different $x_1, x_2 \in R$.

Obviously, Condition $(\tilde{A}_1)$ is less conservative than (A_1), and some general activation functions in conventional neural networks, such as the standard sigmoidal functions, satisfy Condition (A_1) or $(\tilde{A}_1)$. It should be noted that the activation function satisfies Condition (A_1) or $(\tilde{A}_1)$ may be non-differentiable and/or unbounded. Moreover, Condition $(\tilde{A}_1)$ implies that the activation function may be non-monotonic [22, 23].

Definition 1 [11]. The trivial solution of system (3) is said to be globally exponentially stable if, for any solution $y(t,t_0,\phi)$ with the initial condition $\phi \in PC$, there exist constants $\lambda > 0$, $M \geq 1$ such that

$$\|y(t,t_0,\phi)\| \leq M\|\phi\|_r \exp^{-\lambda(t-t_0)}, \quad t \geq t_0. \tag{4}$$

3 Main results

In this section, we derive the main results which ensure the stability of the equilibrium point of the impulsive Hopfield-type neural networks with time-varying delays (1), i.e., the stability of the trivial solution of system (2). We have the following results:

Theorem 1. Assume that (A_1) holds, and there exist n positive numbers $p_1,\ldots,p_n$, such that the following conditions are satisfied for all $i = 1,2,\cdots,n$, and $k \in Z^+$,

(A_2) $-2 < d_{ik} < 0$,

(A_3) There exist two numbers ε_1, $\varepsilon_2 \in [0,1]$ such that

$$\frac{\ln \max\limits_{\{1\leq i\leq n,\, k\in Z^+\}} (1+d_{ik})^2}{\sup\limits_{k\in Z^+} \{t_k - t_{k-1}\}} + \alpha + \frac{\beta}{\max\limits_{\{1\leq i\leq n,\, k\in Z^+\}} (1+d_{ik})^2} < 0$$

where $\alpha = \max\limits_{1\leq i\leq n} \Big(-2c_i + 2a_{ii}^+ k_i + \sum\limits_{j=1, j\neq i}^{n} \Big\{ |a_{ij}k_j|^{2\varepsilon_1} + \frac{p_j}{p_i}|a_{ji}k_i|^{2-2\varepsilon_1} \Big\} + \sum\limits_{j=1}^{N} |b_{ij}l_j|^{2\varepsilon_2} \Big)$,

with $a_{ii}^+ = \max\{a_{ii}, 0\}$ and $\beta = \max\limits_{1\leq i\leq n} \sum\limits_{j=1}^{N} \frac{p_j}{p_i}|b_{ji}l_i|^{2-2\varepsilon_2}$, then the trivial solution of (2) is globally exponentially stable, which implies that the equilibrium point of system (1) is globally exponentially stable.

The detail proof of this result is given in [24]. It is worth mentioning that the result does not require the boundedness and the differentiability of the activation functions, and the differentiability of the time-varying delays. Therefore, Our results have improved and generalized some published results. Moreover, it is shown that an unstable delayed neural networks can be made globally exponentially stable by an appropriate sequence of impulses. This point may be highly important significance in the stability design of neural networks when both delay effect and impulsive effect are taken into consideration. More concretely, for a given delayed neural networks, i.e., the system (1) without impulsive effects, α and β can be derived by simple calculation if $p_k(k=1,\ldots,n)$ are given, then the stability design of the delayed neural networks can be achieved by appropriately choosing the magnitude of impulses d_{ik} and impulsive interval $t_k - t_{k-1}$ such that all the conditions of Theorem 1 are satisfied. In addition, , we can see from (A_3) that the more frequent the impulses are, the more benefit it is to stabilize the underlying delayed neural networks. This is consistent with the intuition cognition.

When the activation function f_r, and g_r satisfy Condition $(\tilde{A}_1)$, we obtain the following theorem.

Theorem 2. Assume that $(\tilde{A}_1)$ holds, and there exist n positive numbers $p_1,\ldots,p_n$, such that the following conditions are satisfied for all $i=1,2,\cdots,n$, and $k \in Z^+$,

$(\tilde{A}_2)$ $-2 < d_{ik} < 0$,

$(\tilde{A}_3)$ There exist two numbers ε_1, $\varepsilon_2 \in [0,1]$ such that

$$\frac{\ln \max\limits_{\{1\le i\le n,\, k\in Z^+\}} (1+d_{ik})^2}{\sup\limits_{k\in Z^+} \{t_k - t_{k-1}\}} + \tilde{\alpha} + \frac{\beta}{\max\limits_{\{1\le i\le n,\, k\in Z^+\}} (1+d_{ik})^2} < 0,$$

where $\tilde{\alpha} = \max\limits_{1\le i\le n}\Big(-2c_i + 2a_{ii}k_i + \sum\limits_{j=1,j\neq i}^{n}\Big\{|a_{ij}k_j|^{2\varepsilon_1} + \frac{p_j}{p_i}|a_{ji}k_i|^{2-2\varepsilon_1}\Big\} + \sum\limits_{j=1}^{N}|b_{ij}l_j|^{2\varepsilon_2}\Big)$, then the trivial solution of (2) is globally exponentially stable, which implies that the equilibrium point of system (1) is globally exponentially stable.

4 Example

In this section, we present a numerical example and its simulation to illustrate that our results can be applied to stabilize the unstable continuous system by using impulses.

Consider the following two-dimensional impulsive neural network with time-varying delays:

$$\begin{cases} \dot{x}_i(t) = -c_i x_i(t) + \sum\limits_{j=1}^{2} a_{ij} f_j(x_j(t)) + \sum\limits_{j=1}^{2} b_{ij} g_j(x_j(t-\tau(t))),\ t \neq t_k, t \geq t_0, \\ \Delta x_i(t)|_{t=t_k} = d_{ik}(x_i(t_k^-) - x_i^*), \quad k \in Z^+, i = 1,2. \end{cases} \tag{5}$$

where

$$C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & -4 \\ -3 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} -2 & -4 \\ -2 & -5 \end{pmatrix}, \tag{6}$$

and $f_j(s) = g_j(s) = \tanh(s)\,(j=1,2)$, $0 \le \tau(t) \le 1.50$. It is easy to know that the point $x^* = (x_1^*, x_2^*)^\top = (6.1363, -1.4318)^\top$ is a equilibrium point of the system (5).

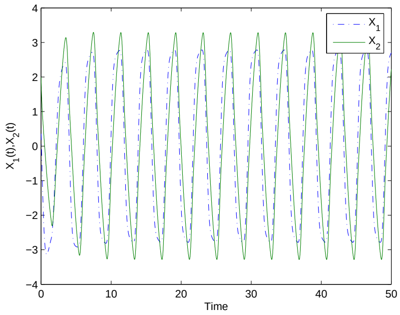

Figure 1: The time response of the state variables for system (5)without impulsive effects.

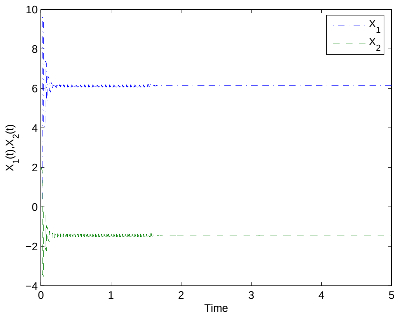

Figure 2: The time response of the state variables for system (5)with $d = -1.60$.

Let $k_1 = k_2 = 1$, $l_1 = l_2 = 1$, $p_1 = p_2 = 1$, and $\varepsilon_1 = \varepsilon_2 = 1/2$, by simple calculation, we can easily get that $\alpha = 14$ and $\beta = 9$. Denote $\tilde{d} = \max_{\{1 \le i \le N,\, k \in Z^+\}} (1 + d_{ik})^2$, then it is easy to verify that if the following condition hold,

$$\sup_{k \in Z^+} \{t_k - t_{k-1}\} < \frac{-\tilde{d} \ln \tilde{d}}{14\tilde{d} + 9}, \quad 0 < \tilde{d} < 1, \tag{7}$$

then all the conditions of Theorem 1 are satisfied, which means the the equilibrium point x^* of system (5) is globally exponentially stable..

In order to illustrate the effectiveness of the above result, for simplicity, we consider the equidistant impulsive interval $t_k - t_{k-1} \equiv \Delta$ and $d_{ik} \equiv d$ for $i = 1, 2$ and $k \in Z^+$. Let $\Delta = 0.02$, Fig.1-Fig.2 are the simulations results corresponding to change process of the state variables of the system (5) with $d = 0$ and $d = -1.60$, respectively. It can be seen that the unstable system (5) without impulsive effects (see Fig.1) can be globally exponentially stabilized by an appropriate sequence of impulses (see Fig.2).

5 Conclusion

In this paper, we have further investigated the stability problem of impulsive neural networks with time-varying delays in the case that the underlying continuous delayed neural networks are unstable. By establishing an impulsive delayed differential inequality, some novel and less conservative criteria for global exponential stability of the equilibrium point of such model have been derived analytically. It is shown that an unstable delayed neural networks can be made globally exponentially stable by an appropriate sequence of impulses. The obtained results do not require the differentiability and/or monotonicity of the activation functions, and the differentiability of the time-varying delays. Our results have improved and generalized some published results and are help to design stability of neural networks when both delay effect and impulsive effect are taken into consideration. An example is also given to show the effectiveness of the new results.

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