The Eighth International Symposium on Operations Research and Its Applications (ISORA'09)
Zhangjiajie, China, September 20–22, 2009

Double Exponential Jump Diffusion Processes and Its Application to Real Options

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Abstract In this paper, we consider optimal stopping problem for double exponential jump diffusion processes. Moreover, we derive the value function of the option to postpone and its optimal boundary. Also some numerical results are presented to demonstrate analytical sensitives of the value function with respect to parameters.

Keywords Double exponential Jump diffusion process; optimal stopping problem

1 Introduction

In option pricing, Jump diffusion model was first introduced by Merton [5]. Mordecki [6] gave the closed form solution under jump diffusion process from the point of view of an optimal stopping problem. Ohnishi [7] discussed an optimal stopping problem with random jumps and derived the value function and the optimal stopping boundary. Kou and Wang [2] have studied the first hitting time for a double exponential jump diffusion process. Moreover, Kou and Wang [3] gave the closed form for the value function of perpetual American put options without dividend and so on.

In this paper, we deal with the option to postpone for double exponential jump diffusion processes and formulate the valuation as an optimal stopping problem. Moreover, we derive the value function of the option to postpone and its optimal boundary.

2 Preliminary

Let $W(t)$ be a standard Brownian motion and $N(t)$ be a Poisson process with the intensity λ. Let J_i denote i.i.d. positive random variables. $Y_i \equiv \log J_i$ has a double exponential distribution and its density function is given by

$$f(y) = p\eta_1 e^{-\eta_1 y} 1_{\{y \geq 0\}} + q\eta_2 e^{\eta_2 y} 1_{\{y<0\}},$$

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where $\eta_1 > 1, \eta_2 > 0$ and $0 \le p, q \le 1$ such that $p + q = 1$. The jump diffusion process $S(t)$ satisfies the stochastic differential equation

$$\frac{dS(t)}{S(t-)} = \mu dt + \kappa dW(t) + d(\sum_{i=1}^{N(t)} (J_i - 1)), \tag{1}$$

where μ and $\kappa > 0$ are constants. Define another probability measure $\tilde{P}$ as

$$\frac{d\tilde{P}}{dP}\Big|_{\mathscr{F}_t} = \exp\left\{-bW(t) - \frac{1}{2}b^2 t\right\},\ b = \frac{\mu - r + d + \lambda\zeta}{\kappa},$$

where r and d are the positive constants. $\mathscr{F}_t = \sigma(W(s), N(s); s \le t, \{J_i\})$ and

$$\zeta = E[J_i] - 1 = \frac{p\eta_1}{\eta_1 - 1} + \frac{q\eta_2}{\eta_2 + 1} - 1.$$

By Girsanov's theorem, $\tilde{W}(t) = W(t) - bt$ is a Brownian motion with respect to $\tilde{P}$.

We can rewrite (1) as

$$\frac{dS(t)}{S(t-)} = (r - d - \lambda\zeta)dt + \kappa d\tilde{W}(t) + d(\sum_{i=1}^{N(t)} (J_i - 1)). \tag{2}$$

Solving (2) gives $S(t) = S(0)\exp X(t)$, where

$$X(t) = \left(r - d - \frac{1}{2}\kappa^2 - \lambda\zeta\right)t + \kappa\tilde{W}(t) + \sum_{i=1}^{N(t)} Y_i.$$

Let $V(v)$ be a function of class C^2. Then the infinitesimal generator $\mathscr{L}$ of the process $S(t)$ is given by

$$\mathscr{L}V(v) = \frac{1}{2}\kappa^2 v^2 V''(v) + (r - d - \lambda\zeta)vV'(v) + \lambda\int_{-\infty}^{\infty} (V(ve^y) - V(v))f(y)dy$$

for all $v > 0$.

Next we introduce the four real numbers $\beta_1, \beta_2, \beta_3$ and β_4. Kou and Wang [2] showed that the equation $G(\theta) = \alpha$ for all $\alpha > 0$ has the solutions $\beta_1,\ \beta_2, -\beta_3$, and $-\beta_4$, where

$$G(\theta) = \theta\left(r - d - \frac{1}{2}\kappa^2 - \lambda\zeta\right) + \frac{1}{2}\theta^2\kappa^2 + \lambda\left(\frac{p\eta_1}{\eta_1 - \theta} + \frac{q\eta_2}{\eta_2 + \theta} - 1\right).$$

And the four solutions satisfy

$$0 < \beta_1 < \eta_1 < \beta_2 < \infty,\ \ 0 < \beta_3 < \eta_2 < \beta_4 < \infty.$$

Let $\mathscr{T}_{0,\infty}$ denote the set of all stopping times with values in the interval $[0,\infty]$. We consider the optimal stopping problem

$$V^*(v) = \sup_{\tau\in\mathscr{T}_{0,\infty}} E[e^{-r\tau}(S(\tau) - I)^+ \mid S(0) = v],\ I > 0$$

where the supremum is taken for all stopping times τ. And the optimal stopping time $\hat{\tau}$ is given by

$$\hat{\tau} = \inf\{t > 0 \mid S(t) \in \mathscr{S}\}$$

where $\mathscr{S} = \{v \mid V^*(v) = (v-I)^+\}$.

We introduce the function $V(v)$ by

$$V(v) = \begin{cases} Av^{\beta_1} + Bv^{\beta_2}, & 0 < v < v_0 \\ v - I, & v \geq v_0. \end{cases}$$

We set $v = e^x$ and $V(v) = V(e^x) \equiv \hat{V}(x)$. We determine the coefficients A_1, A_2 and $v_0 = e^{x_0}$.

By value matching condition, we have

$$Ae^{\beta_1 x_0} + Be^{\beta_2 x_0} = e^{x_0} - I \tag{3}$$

and by smooth pasting condition, we have

$$A\beta_1 e^{\beta_1 x_0} + B\beta_2 e^{\beta_2 x_0} = e^{x_0}. \tag{4}$$

We can get the last condition by using the infinitesimal generator $\hat{\mathscr{L}}$ of the process $X(t)$ given by

$$\mathscr{L}\hat{V}(x) = \frac{1}{2}\kappa^2 \hat{V}''(x) + (r - d - \frac{1}{2}\kappa^2 - \lambda\zeta)\hat{V}'(x) + \lambda \int_{-\infty}^{\infty} (\hat{V}(x+y) - \hat{V}(x)) f(y) dy$$

for all $x > 0$. For $x < x_0$, we obtain

$$\begin{aligned} &\int_{-\infty}^{\infty} \hat{V}(x+y) f(y) dy \\ = \;& \int_0^{x_0 - x} (Ae^{\beta_1(x+y)} + Be^{\beta_2(x+y)})\eta_1 e^{-\eta_1 y} dy + \int_{x_0 - x}^{\infty} e^{x+y}\eta_1 e^{-\eta_1 y} dy \\ = \;& \eta_1 \left(\frac{A}{\eta_1 - \beta_1} e^{\beta_1 x} + \frac{B}{\eta_1 - \beta_2} e^{\beta_2 x} \right) \\ & -\eta_1 e^{-\eta_1(x_0 - x)} \left(\frac{A}{\eta_1 - \beta_1} e^{\beta_1 x_0} + \frac{B}{\eta_1 - \beta_2} e^{\beta_2 x_0} - \frac{e^{x_0}}{\eta_1 - 1} \right). \end{aligned}$$

From this, we obtain

$$\begin{aligned} & (\hat{L} - r)\hat{V}(x) \\ = \;& Ae^{\beta_1 x} \left(\frac{1}{2}\beta_1^2 + \beta_1(r - d - \frac{1}{2}\kappa^2 - \lambda\zeta) \right) + Be^{\beta_2 x} \left(\frac{1}{2}\beta_2^2 + \beta_2(r - d - \frac{1}{2}\kappa^2 - \lambda\zeta) \right) \\ & + \lambda \int_{-\infty}^{\infty} \hat{V}(x+y) f(y) dy - (\lambda + r)\hat{V}(x) \\ = \;& Ae^{\beta_1 x} g(\beta_1) + Be^{\beta_2 x} g(\beta_2) \\ & -\lambda p \eta_1 e^{-\eta_1(x_0 - x)} \left(\frac{A}{\eta_1 - \beta_1} e^{\beta_1 x_0} + \frac{B}{\eta_1 - \beta_2} e^{\beta_2 x_0} - \frac{e^{x_0}}{\eta_1 - 1} \right), \end{aligned}$$

where $g(x) = G(-x) - r$. By Lemma 2.1 in Kou and Wang [2], we have $g(\beta_1) = g(\beta_2) = 0$. Since $(\hat{\mathscr{L}} - r)\hat{V}(x) = 0$ holds, we get the condition

$$\frac{A}{\eta_1 - \beta_1} e^{\beta_1 x_0} + \frac{B}{\eta_1 - \beta_2} e^{\beta_2 x_0} - \frac{e^{x_0}}{\eta_1 - 1} = 0. \tag{5}$$

Lemma 1.
Solving the following equations

$$\begin{aligned} A e^{\beta_1 x_0} + B e^{\beta_2 x_0} &= e^{x_0} - I \\ A\beta_1 e^{\beta_1 x_0} + B\beta_2 e^{\beta_2 x_0} &= e^{x_0} \\ \frac{A}{\eta_1 - \beta_1} e^{\beta_1 x_0} + \frac{B}{\eta_1 - \beta_2} e^{\beta_2 x_0} &= \frac{e^{x_0}}{\eta_1 - 1} \end{aligned}$$

give the solutions

$$\begin{aligned} A &= e^{-\beta_1 x_0} \frac{\beta_2 - 1}{\beta_2 - \beta_1} \left(e^{x_0} - \frac{\beta_2}{\beta_2 - 1} I \right), \\ B &= e^{-\beta_2 x_0} \frac{\beta_1 - 1}{\beta_2 - \beta_1} \left(\frac{\beta_1}{\beta_1 - 1} I - e^{x_0} \right). \end{aligned}$$

3 Main result

In this section we give the main theorem. In order to prove it, we needs the following lemmas.

Lemma 2.
Assume that a function $V(v)$ has the following properties.

1. $(\mathscr{L} - r)V(v) \le 0$, *for* $v > v_0$.
2. *It holds* $(\mathscr{L} - r)V(v) = 0$ *and* $V(x)$ *satisfies* $V(v) > (v - I)^+$ *for* $0 < v < v_0$.
3. *At* $v = v_0$ *we have* $V'(v_0-) = V'(v_0+)$.

Then, V is the value function of the option to postpone , i.e., *$V^* = V$ holds. The optimal exercise region is the interval* $[v_0, \infty)$.

In what follows we will explore the properties of the function $V(v)$ in the above lemma.

Lemma 3.
For $v > v_0$ the function $V(v)$ satisfies

$$(\mathscr{L} - r)V(v) \le 0.$$

Lemma 4.
It holds $(\mathscr{L} - r)V(v) = 0$ *and* $V(v)$ *satisfies* $V(v) > (v - I)^+$ *for* $0 < v < v_0$.

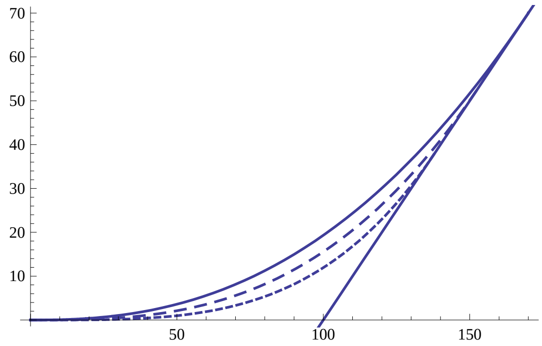

Figure 1: The value function

Lemma 5.
At $v = v_0$ we have $V'(v_0-) = V'(v_0+)$.

Theorem 6.
Let $V^*(v)$ denote the value function of the option to postpone. Then $V^*(v)$ is given by

$$V^*(v) = \begin{cases} A(v_0)v^{\beta_1} + B(v_0)v^{\beta_2}, & 0 \le v \le v_0 \\ v - I, & v \ge v_0 \end{cases} \tag{6}$$

and the optimal stopping times is given by

$$\hat{\tau} = \inf\{t > 0 \mid S(t) \ge v_0\},$$

where the optimal boundary v_0 is

$$v_0 = \frac{\eta_1 - 1}{\eta_1} \frac{\beta_1}{\beta_1 - 1} \frac{\beta_2}{\beta_2 - 1} I,$$

the coefficients A and B are

$$\begin{aligned} A(v_0) &= v_0^{-\beta_1} \frac{\beta_2 - 1}{\beta_2 - \beta_1} \left(v_0 - \frac{\beta_2}{\beta_2 - 1} I \right), \\ B(v_0) &= v_0^{-\beta_2} \frac{\beta_1 - 1}{\beta_2 - \beta_1} \left(\frac{\beta_1}{\beta_1 - 1} I - v_0 \right). \end{aligned}$$

Moreover, the value function $V^*(v)$ is also represented by

$$V^*(v) = \tilde{E}[\int_0^\infty e^{-\alpha t}(r - \mathscr{L})V(S(t))dt].$$

4 Numerical example

In this section we present numerical example. We set $r = 0.1$, $=0.09$, $\kappa = 0.2$, $\eta_1 = 50$, $\lambda = 3$ and $I = 100$. By using these parameters, one has $\beta_1 = 2.426$ and $\beta_2 = 51.480$. Figure 1 demonstrates the value function of the option to postpone($\kappa = 0.1, 0.15, 0.2$). From this figure, we can recognize that $V^*(v)$ is convex and increasing in v.

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