The Eighth International Symposium on Operations Research and Its Applications (ISORA'09)
Zhangjiajie, China, September 20–22, 2009

Performance Analysis of Geom^X/G/1 Queue with Exhaustive Service Rule and Multiple Vacations

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Abstract In this paper, a new discrete-time Geom^X/G/1 queue model with multiple vacations is analyzed. The Probability Generating Function (P.G.F.) of the queue length is obtained by using the method of an embedded Markov chain, and the mean of the queue length is obtained by using L'Hospital rule. Then the P.G.F. of the busy period is derived, and the probabilities for the system being in a busy state or in a vacation state are also derived. Moreover, the P.G.F. of the waiting time is derived based on the independence between the arrival process and the waiting time. Finally, some numerical results are shown to compare the means of the queue length and the waiting time in special cases.

Keywords Geom^X/G/1 queue, multiple vacations, embedded Markov chain, exhaustive service rule, performance analysis.

1 Introduction

In the classical M^X/G/1 queue, many M^X/G/1 queues with vacation policy have been studied by many researchers, and the Probability Generating Function (P.G.F.) of the queue length and the Laplace-Stieltjes Transform (LST) of the waiting time have been obtained [1]-[3]. For example, Hou and Lu studied an $M_1^{X_1}$, $M_2^{X_2}$/G/1 queue with single vacation, obtained the P.G.F. of the queue length and the LST of the waiting time in [4]. Kawasaki and Takahashi analyzed the waiting time of a customer of an M^X/G/1 queue with/without vacations under a random order of service discipline in [5]. Thomo studied an M^X/G/1 queue with balking and multiple vacations by giving the P.G.F. of the queue length and the LST of the waiting time in [6]. Lee, et al. analyzed an M^X/G/1 queue with N-policy and multiple vacations and obtained some performance measures of the system in [7].

However, the studies about Geom^X/G/1 queues are less numerous than those relating to M^X/G/1 queues. Therefore, we analyze a new discrete-time Geom^X/G/1 queue with exhaustive service rule and multiple vacations, based on the results of some former studies. We also obtain the P.G.F. of the stationary queue length, the LST of the waiting time and the probability of system states. We give some numerical results for comparing these performance measures. So the queue model enriches the theory of the Geom^X/G/1 queue with vacation, and involves some special case queue modeling.

The paper is organized as follow. In section 2, we give the description of the new model and its system parameters in detail. In section 3, we embed a Markov chain, and show the necessary sufficient condition for positive recurrence. In section 4, we derive the P.G.F. of the stationary queue length, the LST of the waiting time, and stochastic decomposition results of stationary measures. In section 5, we discuss some numerical results. Concluding remarks are given in section 6.

2 System Model

Based on the classical Geom/G/1 queue model, we study the queue model with an exhaustive service rule and multiple vacations in [8], [9]. The service strategy can be described as follows. Once there is no customer in the system, the server enters into a vacation period of random length V. If there are some customers waiting at the vacation completion instant, the server will start a new busy period until the system becomes empty again, otherwise the server will take consecutive vacations according to the assistant work. The possible number of vacations is from 1 to ∞. The system will continually repeat the above processes. The queue model is denoted by Geom^X/G/1 (E, MV), where E means exhaustive service rule, MV means multiple vacations.

The basic assumptions of the new model presented in this paper are given as follows:
(1) We assume that the system states are in the discrete time instants. We assume that customer arrivals can only occur at discrete time instants $t = n^-, n = 0, 1, ...$, the service starts and ends can only occur at discrete time instants $t = n^+, n = 1, 2, ...$ The model is called a late arrival system. The inter-arrival time of a batch, denoted by J_n, is supposed to be an independently identically distributed (i.i.d.) discrete random variable following a geometric distribution with the parameter p $(0 < p < 1)$. We can write the probability distribution of J_n as follows:

$$P\{J_n = k\} = p\bar{p}^{k-1}, \quad k = 1, 2, ...$$

where $\bar{p} = 1 - p$.
(2) The service time sequence $\{B_n, n \geq 1\}$ is an i.i.d. discrete random variable sequence with a general distribution. The probability distribution g_k and the Probability Generating Function (P.G.F.) $G(z)$ of g_k are given as follows:

$$P\{B_n = k\} = g_k, \quad k \geq 1, \quad G(z) = \sum_{k=1}^{\infty} g_k z^k.$$

Let $E[B_n] = \dfrac{1}{\mu}$ be the mean of B_n. The variance $D(B_n) = \sigma^2$ and the third origin moment $E[B_n^3]$ of B_n exist and are limited.
(3) In this system model, the time axis is discrete-time into sequence of equal length called slots. We denote by X the number of customers arriving during a single slot, the P.G.F. of X is $R(z)$. The mean and variance of X are $E[X] = r$ and $D(X) = \sigma_r^2$, respectively.
(4) The time length V_n $(n \geq 1)$ of a vacation is a positive i.i.d. discrete random variable with general probability distribution v_j and the P.G.F. $V(z)$ given by

$$P\{V_n = j\} = v_j, \quad j \geq 1, \quad V(z) = \sum_{j=1}^{\infty} v_j z^j.$$

(5) Suppose that there is a single server in this system, and its buffer capacity is infinite. $\{J_n, n \geq 1\}$, $\{B_n, n \geq 1\}$, $\{V_n, n \geq 1\}$ and X are mutually independent. The service order is First-Come First-Served (FCFS).

3 Embedded Markov Chain

Let L_n represent the number of customers in the system left behind by the nth departing customer and let Y_n represent the number of customers arriving in the service period of the nth customer. Let $N(t)$ be the number of customers arriving during the interval $[0,t]$, i.e. $Y_n = N(B_n)$. Let Q_b be the number of customers in the system at the start instant of a busy period. The probability distribution and P.G.F. of Q_b are represented by $b_i,\ i \geq 1$ and $Q_b(z)$. We define a function as follows:

$$\varepsilon(x) = \begin{cases} 1, & x > 0, \\ 0, & x \leq 0. \end{cases}$$

Since the basic assumptions of the system model as Y_{n+1}, L_n and Q_b are mutually independent, we have that

$$L_{n+1} = L_n - \varepsilon(L_n) + Y_{n+1} + (Q_b - 1)\varepsilon(1 - L_n). \tag{1}$$

Theorem 1.
Random sequence $\{L_n,\ n \geq 1\}$ is a homogeneous embedded Markov chain.

Theorem 2.
For the embedded Markov chain $\{L_n,\ n \geq 1\}$ of Geom^X/G/1 (E, MV), the necessary and sufficient condition of positive recurrence is the traffic intensity $\rho = \dfrac{pr}{\mu} < 1$, where p is distribution parameter of the inter-arrival time of a batch and r is the mean of X presented in Section 2.
The proofs of theorems 1 and 2 are omitted.

Since $Y_{n+1} = X_1 + X_2 + ... + X_{N(B_{n+1})}$, $(X_0 \equiv 0)$, we have that

$$E[Y_{n+1}] = E[X_1]E[N(B_{n+1})] = \frac{pr}{\mu} = \rho.$$

We denote by $C_j^{(v)}$ the probability that there are j batch arrivals during a vacation. Then we have that $C_j^{(v)} = \sum\limits_{r=j}^{\infty} v_r \binom{r}{j} p^j \bar{p}^{r-j},\ j \geq 0$ and

$$P(Q_b = k) = \sum_{i=1}^{k} \frac{C_i^{(v)}}{1 - V(\bar{p})} P(X_1 + \cdots + X_i = k), \quad k \geq 1,$$

$$Q_b(z) = \sum_{k=1}^{\infty} z^k \sum_{i=1}^{k} \frac{C_i^{(v)}}{1 - V(\bar{p})} P(X_1 + \cdots + X_i = k) = \frac{V(1 - p(1 - R(z))) - V(\bar{p})}{1 - V(\bar{p})},$$

$$E[Q_b] = Q_b'(z)|_{z=1} = \frac{prE[V]}{1 - V(\bar{p})}$$

where $V(\bar{p}) = \sum\limits_{j=1}^{\infty} v_j (\bar{p})^j$ and $V(1 - p(1 - R(z))) = \sum\limits_{j=1}^{\infty} v_j (1 - p(1 - R(z)))^j$.

4 Performance Analysis of Queue System

4.1 Stationary Queue Length

If $\rho = \dfrac{pr}{\mu} < 1$, let $\lim_{n\to\infty} E[L_{n+1}] = \lim_{n\to\infty} E[L_n] = E[L]$, $\lim_{n\to\infty} E[Y_{n+1}] = E[Y] = E[X]E[N(B)]$, where L is the limits of L_n, Y is the limits of Y_n, X is the number of customers arriving during a single slot, $N(B)$ is the number of customers arriving during a busy period. Taking the mean of the two sides of Eq. (1), we obtain the equation as follows:

$$E[\varepsilon(L_n)] = E[Y] + E[Q_b - 1]E[\varepsilon(1-L_n)].$$

Since we have that

$$P\{L>0\} = \rho + \left(\frac{prE[V]}{1-V(\bar{p})} - 1\right) P\{L=0\}$$

therefore

$$P\{L=0\} = \frac{(1-\rho)(1-V(\bar{p}))}{prE[V]}, \quad P\{L>0\} = \frac{prE[V]-(1-\rho)(1-V(\bar{p}))}{prE[V]}$$

where $E[V]$ is the mean of V.

Taking the square of the two sides of Eq. (1), we have that

$$\begin{aligned} L_{n+1}^2 = {} & L_n^2 + \varepsilon(L_n) + Y_{n+1}^2 + (Q_b-1)^2\varepsilon(1-L_n) \\ & -2L_n + 2L_nY_{n+1} - 2Y_{n+1}\varepsilon(L_n) + 2Y_{n+1}(Q_b-1)\varepsilon(1-L_n) \end{aligned} \tag{2}$$

where

$$E[Y_{n+1}^2] = E[N^2(B_{n+1})] = r\rho + \rho^2 + p^2r^2\sigma^2 + \frac{p}{\mu}\sigma_r^2,$$

$$\begin{aligned} E[(Q_b-1)^2\varepsilon(1-L_n)] &= P(L_n=0)(E[Q_b^2] - 2E[Q_b] + 1) \\ &= \frac{(1-\rho)}{prE[V]}\left(p^2E[V^2]r^2 + (1-V(\bar{p})) + pE[V](\sigma_r^2 + r^2 - r)\right), \end{aligned}$$

$$E[2L_nY_{n+1}] = 2E[Y_{n+1}]E[L_n] = 2\rho E[L_n],$$

$$E[2Y_{n+1}\varepsilon(L_n)] = 2\rho P(L_n>0) = \frac{2\rho(prE[V]-(1-\rho)(1-V(\bar{p})))}{prE[V]},$$

$$E[2Y_{n+1}(Q_b-1)\varepsilon(1-L_n)] = 2\rho(1-\rho)\left(1 - \frac{1-V(\bar{p})}{prE[V]}\right).$$

Taking the mean of the two sides of Eq. (2), we obtain the equation as follows:

$$\begin{aligned}0 = &\frac{prE[V]-(1-\rho)(1-V(\bar{p}))}{prE[V]}+r\rho+\rho^2+p^2r^2\sigma^2\\ &+\frac{p}{\mu}\sigma_r^2+\frac{(1-\rho)}{prE[V]}(p^2E[V^2]r^2+pE[V](\sigma_r^2+r^2-2r)\\ &+(1-V(\bar{p})))-2E[L]+2\rho E[L]-\frac{2\rho(prE[V]-(1-\rho)(1-V(\bar{p})))}{prE[V]}\\ &+2\rho(1-\rho)\left(1-\frac{1-V(\bar{p})}{prE[V]}\right).\end{aligned}$$

Therefore, we obtain that

$$E[L]=\rho+\frac{p^2r^2\sigma^2+\rho^2}{2(1-\rho)}+\frac{\sigma_r^2+\rho^2-r}{2r(1-\rho)}+\frac{prE[V^2]}{2E[V]}. \tag{3}$$

Suppose that the service order of the batch is FCFS, and the service order of customers in the same batch is random. Let $L_n^{(v)}(z)$ be the P.G.F of L_n, and $L_v(z)=\lim_{n\to\infty}L_n^{(v)}(z)$.

Theorem 3.
If $\rho<1$, the stationary queue length L_v in Geom^X/G/1 (E, MV) queue can be decomposed into two independent random variables L and L_d, then we have

$$L_v = L + L_d$$

where L is the stationary queue length in the classical Geom^X /G/1 queue. The P.G.F. $L(z)$ of L is given by

$$L(z)=\frac{(1-\rho)(R(z)-1)G(1-p(1-R(z)))}{rz-rG(1-p(1-R(z)))}$$

where $R(z)$ is the P.G.F. of X and $G(z)$ is the P.G.F. of the service time B_n defined in Section 2, $G(1-p(1-R(z)))=\sum_{k=1}^{\infty}g_k(1-p(1-R(z)))^k$.
L_d is an additional queue length. The P.G.F. $L_d(z)$ of L_d is given by

$$L_d(z)=\frac{1-V(1-p(1-R(z)))}{pE[V](1-R(z))}.$$

Proof. Taking the P.G.F. of the two sides of Eq. (1), and letting $n\to\infty$, we obtain the equation as follows:

$$\begin{aligned}L_v(z) &= \lim_{n\to\infty}L_n^{(v)}(z)\\ &= E[z^Y]\lim_{n\to\infty}E[z^{(L_n-\varepsilon(L_n)+(Q_b-1)\varepsilon(1-L_n))}]\\ &= G(1-p(1-R(z)))\lim_{n\to\infty}\sum_{k=0}^{\infty}E[z^{(k-\varepsilon(k)+(Q_b-1)\varepsilon(1-k))}]P(L_n=k)\\ &= G(1-p(1-R(z)))\left(\frac{(1-\rho)(1-V(\bar{p}))}{prE[V]}\frac{V(1-p(1-R(z)))-V(\bar{p})}{z(1-V(\bar{p}))}\right.\\ &\quad\left.+\frac{1}{z}\left(L_v(z)-\frac{(1-\rho)(1-V(\bar{p}))}{prE[V]}\right)\right).\end{aligned} \tag{4}$$

Combining and arranging Eq. (4), we obtain the P.G.F. $L_v(z)$ as follows:

$$L_v(z) = \frac{(1-\rho)(R(z)-1)G(1-p(1-R(z)))}{rz - rG(1-p(1-R(z)))} \frac{1-V(1-p(1-R(z)))}{pE[V](1-R(z))}. \quad (5)$$

Therefore, the P.G.F. $L_d(z)$ and the mean $E[L_d]$ of the additional queue length L_d are given by

$$L_d(z) = \frac{1-V(1-p(1-R(z)))}{pE[V](1-R(z))}, \quad E[L_d] = \frac{prE[V^2]}{2E[V]}. \qquad \square$$

4.2 Busy Period and Busy Cycle

For a Geom^X/G/1 (E, MV) queue, the busy period is considered to be a time interval, in which the system serves all Q_b customers arrived in the system until the system becomes empty under the condition that when the system comes back from a vacation state, there are Q_b customers waiting in the system. Let Θ denote the total busy period of the queue model when there are Q_b customers arrived in the system. So Θ will be the busy period of Geom^X/G/1 (E, MV) queue. Let θ denote the time interval in which the system serves one customer and all customers arrived in the system when the system serves this customer. Let $\theta(z)$ and $\Theta(z)$ be the P.G.Fs. of θ and Θ, we then have that

$$\Theta = \sum_{i=1}^{Q_b} \theta_i$$

where $\theta_1, \theta_2, \theta_3, \ldots$ are mutually independent, and they have the same distribution as θ. We have that

$$\Theta(z) = Q_b(\theta(z)) = \frac{V(1-p(1-R(\theta(z)))) - V(\bar{p})}{1-V(\bar{p})}.$$

Since

$$\theta = B + \Theta_1 + \Theta_2 + \ldots + \Theta_{N(B)}, \quad \Theta = U + \Theta_1 + \Theta_2 + \ldots + \Theta_{N(U)}$$

where $\Theta_1, \Theta_2, \ldots$ are mutually independent, and have the same distribution as Θ. $U = \sum_{i=1}^{Q_b} B_i$, $B_1, B_2, \ldots$ are mutually independent, and have the same distribution as B, where B is the busy period of Geom/G/1 queue and $N(U)$ is the number of customers arriving during the interval $[0, U]$.

$$\theta(z) = E[z^\theta] = G\left(z - pz\frac{1-V(1-p(1-R(\theta(z))))}{1-V(\bar{p})}\right),$$

therefore

$$\Theta(z) = Q_b(\theta(z)) = \frac{V(1-p(1-R(B(z-pz(1-\Theta(z)))))) - V(\bar{p})}{1-V(\bar{p})}.$$

Since

$$E[U] = E\left[\sum_{i=1}^{Q_b} B_i\right] = E[Q_b]E[B] = \frac{prE[V]}{1-V(\bar{p})}\frac{1}{\mu} = \frac{\rho E[V]}{1-V(\bar{p})},$$

$$D(U) = \frac{1}{1-V(\bar{p})}\left(prE[V]\sigma^2 + \rho^2 E[V^2] + \frac{p}{\mu^2}E[V]\sigma_r^2 + \frac{r\rho}{\mu}E[V]\right) - \left(\frac{\rho E[V]}{1-V(\bar{p})}\right)^2,$$

$$E[N(U)] = E\left[N\left(\sum_{i=1}^{Q_b} B_i\right)\right] = \lambda E\left[\sum_{i=1}^{Q_b} B_i\right] = \frac{p\rho E[V]}{1-V(\bar{p})},$$

$$E[\Theta] = E[U] + E\left[\sum_{i=1}^{N(U)} \Theta_i\right] = \frac{\rho E[V]}{1-V(\bar{p})} + \frac{p\rho E[V]}{1-V(\bar{p})}E[\Theta]$$

where $E[U]$ and $D(U)$ are the mean and the variance of U, $E[N(U)]$ is the mean of $N(U)$ and $E[Q]$ is the mean of Q_b. With $E[U]$, $D(U)$ and $E[N(U)]$, we have

$$E[\Theta] = \frac{\rho E[V]}{1-V(\bar{p}) - p\rho E[V]}, \quad E[\theta] = \frac{E[\Theta]}{E[Q_b]} = \frac{1-V(\bar{p})}{\mu(1-V(\bar{p}) - p\rho E[V])}. \quad (6)$$

Define the busy cycle B_c as the time period between two consecutive busy period ending instants, K as the vacation times in one whole vacation period, V_G as the total length of K consecutive vacations, and J_n as inter-arrival time. We have that

$$P\{K = j\} = P\{V^{(j-1)} < J_n < V^{(j)}\} = (V(\bar{p}))^{j-1}(1-V(\bar{p})), \quad j \geq 1,$$

$$V_G = \sum_{j=1}^{\infty}(1-V(\bar{p}))(V(\bar{p}))^{j-1}V^{(j)},$$

therefore, the mean total length of a vacation can be obtained as follows:

$$E[V_G] = \sum_{j=1}^{\infty}(1-V(\bar{p}))(V(\bar{p}))^{j-1}jE[V] = \frac{E[V]}{1-V(\bar{p})}.$$

The mean busy cycle $E[B_c]$ is given as follows:

$$E[B_c] = E[\Theta] + E[V_G] = \frac{E[V]((1+\rho)(1-V(\bar{p})) - p\rho E[V])}{(1-V(\bar{p}))(1-V(\bar{p}) - p\rho E[V])}.$$

Let p_B and p_V be the probabilities of the server being in a busy state and a vacation state, respectively. We have that

$$p_B = \frac{\rho(1-V(\bar{p}))}{(1+\rho)(1-V(\bar{p})) - p\rho E[V]}, \quad p_V = \frac{1-V(\bar{p}) - p\rho E[V]}{(1+\rho)(1-V(\bar{p})) - p\rho E[V]}.$$

4.3 Stationary Waiting Time

Theorem 4.

If $\rho < 1$, the stationary waiting time W_F in a Geom^X/G/1 (E, MV) queue can be decomposed into two independent random variables W and W_d. Then we have

$$W_F = W + W_d$$

where W is the stationary queue length in a classical Geom^X/G/1 queue, the P.G.F. $W(z)$ of W is given by

$$W(z) = \frac{(1-\rho)(1-z)(1-R(G(z)))}{r(1-z-p(1-R(G(z))))(1-G(z))}$$

where $R(G(z))$ is the value of $R(z)$ at $G(z)$ point, $G(z)$ is the P.G.F. of the service time. W_d is the additional delay. The P.G.F. $W_d(z)$ of the additional delay W_d is given by

$$W_d(z) = \frac{1-V(z)}{E[V](1-z)}.$$

Proof: We denote by W_F the waiting time of a customer for the queue system with FCFS order. W_F contains two parts: one part is the waiting time W_1 of the batch in which the customer is, the other part is the waiting time W_2 of the customer in its batch. Let $W_F(z), W_1(z)$ and $W_2(z)$ be the P.G.Fs. of W_F, W_1 and W_2, respectively. Since W_1 and W_2 are mutually independent, we have that

$$W_F(z) = W_1(z)W_2(z). \tag{7}$$

Let the waiting time of a batch of customers be $H = \sum_{i=1}^{X} B_i$, then the P.G.F. $H(z)$ of H is given by

$$H(z) = R(G(z)). \tag{8}$$

Substituting $H(z)$ into $G(z)$ in the P.G.F. of the waiting time for Geom/G/1 (E, MV) queue [9], we have that

$$W_1(z) = \frac{(1-\rho)(1-V(z))}{E[V](1-z-p(1-R(G(z))))}. \tag{9}$$

Deriving the two sides of Eq. (9) and using L'Hospital rule, we obtain the mean $E[W_1]$ of W_1 as follows:

$$E[W_1] = \frac{pE[H^2]}{2(1-\rho)} + \frac{E[V^2]}{2E[V]} = \frac{p(r\mu^2\sigma^2+\sigma_r^2+r^2)}{2\mu^2(1-\rho)} + \frac{E[V^2]}{2E[V]}.$$

Let $[A]$ represent an arbitrary customer in a batch of customers, so customer $[A]$ can appear in any position of X positions. And the service time of a batch of customers is $H = \sum_{i=1}^{X} B_i$. If any one instant is chosen in H, the instant must be in a certain service time B by probability 1, so the age H_- of H is equal to the sum of W_2 and age B_- of B, i.e., $H_- = B_- + W_2$.

Since B_- and W_2 are mutually independent, $H(z) = R(G(z))$, $E[H] = \dfrac{r}{\mu}$ and $D(H) = r\sigma^2 + \dfrac{\sigma_r^2}{\mu^2}$, we have, $H_-(z) = B_-(z)W_2(z)$, then

$$\frac{1-H(z)}{E[H](1-z)} = \frac{1-G(z)}{E[B](1-z)}W_2(z). \tag{10}$$

Simplifying Eq. (10), we have that

$$W_2(z) = \frac{1-H(z)}{1-G(z)}\frac{E[B]}{E[H]} = \frac{1-R(G(z))}{r(1-G(z))}. \tag{11}$$

Deriving the two sides of Eq. (11) and using L'Hospital rule, we obtain the mean $E[W_2]$ of W_2 as follows:

$$E[W_2] = E[H_-] - E[B_-] = \frac{E[H^2]}{2E[H]} - \frac{E[B^2]}{2E[B]} = \frac{\sigma_r^2 + r^2 - r}{2r\mu}.$$

Finally, combining Eqs. (7), (9) and (11), we obtain the P.G.F. $W_F(z)$ and the mean $E[W_F]$ of W_F as follows:

$$W_F(z) = \frac{(1-\rho)(1-V(\bar{p}))(1-R(G(z)))}{rE[V](1-z-p(1-R(G(z))))((1-G(z)))}, \tag{12}$$

$$E[W_F] = \frac{pr^2\mu^2\sigma^2 + \mu\sigma_r^2 + r\mu(r+\rho-1)}{2r\mu^2(1-\rho)} + \frac{E[V^2]}{2E[V]}. \tag{13}$$

5 Numerical Results

In this section, we present some numerical results that provide insight into the system behavior. Using the equations presented in Section 4, we can numerically compare the performance measures of the systems for the three different cases of Geom^X/G/1 (E, MV) queue models with $\theta = 0.25$, $\theta = 0.10$, $\theta = 0.05$, where θ is the parameter of geometric distribution which V follows. Here we assume that the service time S and the time length V of a vacation follow geometric distributions, i.e., S follows a geometric distribution with parameter $\mu = 1/3$. V follows another geometric distribution with parameter θ. By using Eqs. (3) and (13), we can derive the mean queue length $E[L]$ and the waiting time $E[W_F]$. Suppose that the traffic intensity ρ range is from 0.1 to 0.9.

Figure 1 shows the mean queue length $E[L]$ as a function of the traffic intensity ρ with three cases of θ, i.e., $\theta = 0.25$, $\theta = 0.10$, $\theta = 0.05$, where $X = 1/3$. We can find that when ρ increases, $E[L]$ increases to a high level for all the cases. This is because the larger the traffic intensity ρ is, the higher the possibility that there will be customers arriving during the server cycle will be. And we also note that the mean queue length $E[L]$ of Geom^X/G/1 (E, MV) queue with $\theta = 0.05$ is larger than that of Geom^X/G/1 (E, MV) queue with $\theta = 0.25$. This is because the longer the length of the vacation times is, the larger the mean queue length $E[L]$ will be.

Figure 2 shows how the mean waiting time $E[W_F]$ changes with the traffic intensity ρ for the three different cases of θ, i.e., $\theta = 0.25$, $\theta = 0.10$, $\theta = 0.05$, where $X = 1/3$. We can find that when ρ increases, $E[W_F]$ increases to a high level. This is because the larger the traffic intensity ρ is, the higher the possibility that there will be customers arriving during the server cycle will be, so the mean waiting time will be larger. We also note that the mean waiting time $E[W_F]$ of Geom^X/G/1 (E, MV) queue with $\theta = 0.05$ is larger than that of Geom^X/G/1 (E, MV) queue with $\theta = 0.25$. This is because the longer the length of the vacation time is, the greater the mean waiting time $E[W_F]$ will be.

Figure 3 shows the mean queue length $E[L]$ as a function of the traffic intensity ρ with three cases of X. Namely, in the first case, X follows a geometric distribution with

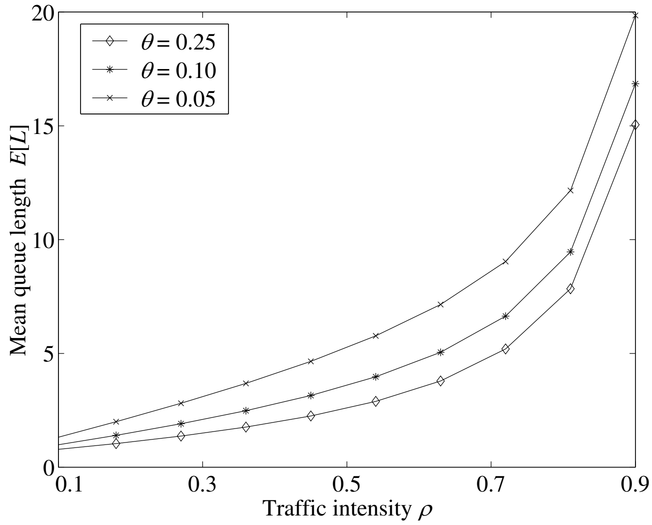

Figure 1: Mean queue length $E[L]$ versus traffic intensity ρ.

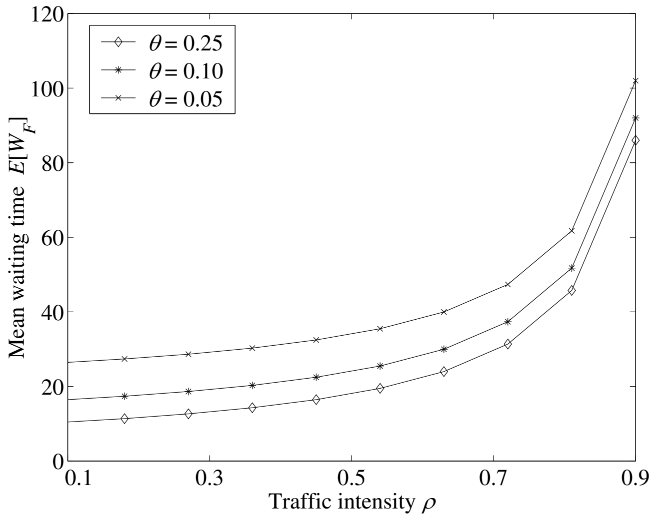

Figure 2: Mean waiting time $E[W_F]$ versus traffic intensity ρ.

parameter $X = 1/3$. In the second case, X follows a degenerate distribution, and in the third case, X follows a Poisson distribution with parameter $\lambda = 3$, where $\theta = 0.25$. We can find that when ρ increases, $E[L]$ increases to a high level in all cases. This is because the larger the traffic intensity ρ is, the higher the possibility that there will be customers arriving during the server cycle will be. We also note that the mean queue length $E[L]$ of a Geom^X/G/1 (E, MV) queue in which X follows degenerate distribution is larger than that of a Geom^X/G/1 (E, MV) queue in which X follows Poisson distribution. This is because

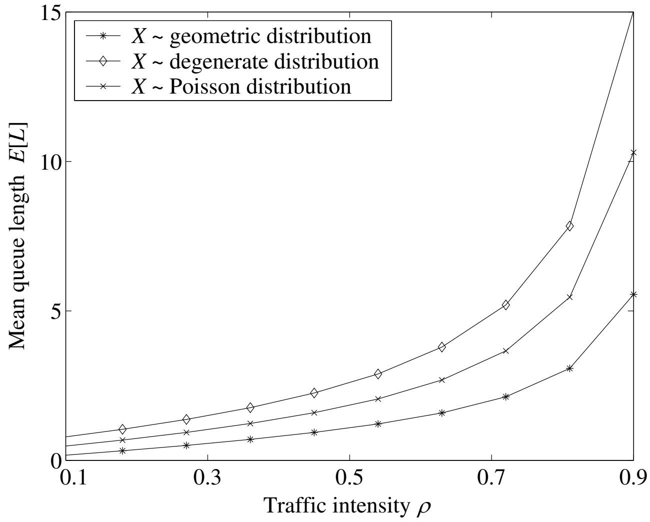

Figure 3: Mean queue length $E[L]$ versus traffic intensity ρ.

the higher the possibility that there are customers arriving during the server cycle, the larger the mean queue length $E[L]$ in the second case will be.

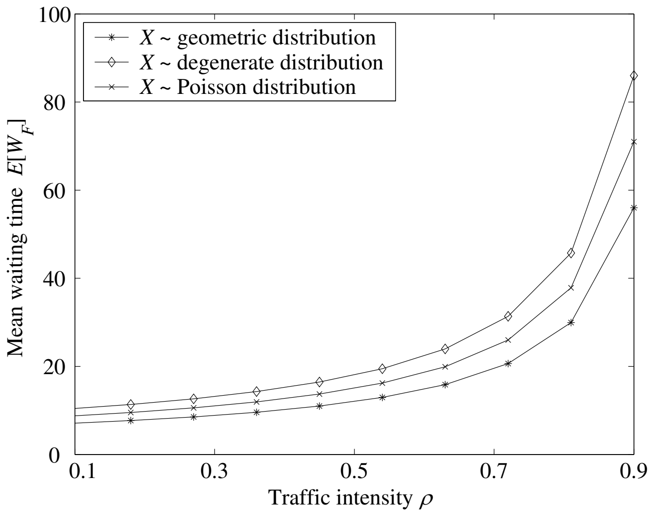

Figure 4: Mean waiting time $E[W_F]$ versus traffic intensity ρ.

Figure 4 shows how the mean waiting time $E[W_F]$ changes with the traffic intensity ρ with three cases of X. Namely, in the first case, X follows a geometric distribution with parameter $X = 1/3$. In the second case, X follows a degenerate distribution, and in the third case, X follows a Poisson distribution with parameter $\lambda = 3$, where $\theta = 0.25$.

We can find that when ρ increases, $E[W_F]$ increases to a high level. This is because the greater the traffic intensity ρ is, the higher the possibility that there will be customers arriving during the server cycle will be, so the mean waiting time will be greater. We also note that the mean waiting time $E[W_F]$ of a Geom^X/G/1 (E, MV) queue in which X follows degenerate distribution is larger than that of a Geom^X/G/1 (E, MV) queue in which X follows Poisson distribution. This is because the higher the possibility that there are customers arriving during the server cycle, the larger the mean waiting time $E[W_F]$ in the second case will be.

6 Conclusion

In this paper, we presented a detailed description of a new discrete-time Geom^X/ G/1 queue model with multiple vacations. By using the method of an embedded Markov chain, we derived the P.G.Fs. of the queue length and the customers waiting time. Furthermore, we presented the stochastic decompositions for the additional queue length and the additional delay. We obtained the probabilities for the system being in a busy state or in a vacation state, respectively. Finally, we gave some numerical results to compare the means of the queue length and the waiting time in special cases. This model is an extension of a Geom/G/1 queue with multiple vacations.

Acknowledgements

This work was supported in part by Hebei Province (No. F2009000475, F2008000864) and the Natural Science Foundations of China (No. 10671170), and was supported in part by GRANT-IN-AID FOR SCIENTIFIC RESEARCH (No. 21500086) and the Hirao Taro Foundation of KUAAR, Japan.

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