The Eighth International Symposium on Operations Research and Its Applications (ISORA'09)
Zhangjiajie, China, September 20–22, 2009

Integrated Convex Optimization in Banach Space and its Application to Best Simultaneous Approximation*

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Abstract Concept of locally ε-maximal is proposed and employed to explored the optimization problem of a set of convex functions on a Banach space in this paper. Generalized saddle point (GSP) is also used to study the optimization problem , which is equivalent to the optimal solution under the condition of locally ε-maximal. An application of the optimization theory to best simultaneous approximation is presented in the paper.

Keywords Convex optimization; Best approximation; Generalized saddle point; Banach space

1 Introduction

It is well known that convex optimization in Banach space has been much interested by people because of its wide range of application in mathematics and engineering. The book written by Th. Precupanu and Viorel Barbu [9] shows that convexity is very essential and useful in optimization theories. The characterization theory of best approximation, collected in a book written by Ivan singer [2], can be generalized to convex optimization, which shows the relationship between best approximation and convex optimization in locally convex space using subdifferentials and directional derivatives. Nonlinear optimization in normed linear spaces has been discussed by [14], but the relationship between best approximation and convex optimization was not in-depth study. Concept of generalized saddle point (GSP) was introduced and used to study optimization problem on a *Banach* space, and its application to best Simultaneous approximation is investigated, that the Kolmogorov condition, an important tool of best approximation, is equivalent to GSP [17].

Based on the study in [17, 18], the research on integrated optimization in *Banach* space is furthered in this paper. Concept of locally-ε-maximal is proposed and employed to analysis the optimization problem of a set of convex functions, under which generalized saddle point(GSP), Fréchet differentiable and Gateaux derivable are equivalent to optimal solution respectively. An application of the optimal theory is explored in best

*This work is partially supported by the Research Fund for Newly-Introduced Teacher in BJFU

simultaneous approximation, and the equivalences in a concise format are given in the research.

The rest of this paper is organized as follows. Concept of locally ε-maximal is proposed in section 2 and Fréchet differentiable and Gateaux derivable are employed to discuss the optimization problem, which displays the concept plays an important role in discussion. The GSP [17] is employed to explored equivalences between optimal solution and GSP, Fréchet differentiable and Gateaux derivable in section 3. An application of the results presented in the section 2 and 3 to best approximation is given in Section 4.

2 Integrated Optimization

Assume X be a Banach space, X^* be the dual space of X, namely, the set of all the linear functional on X, and H be a set of convex real-valued functions ϕs on X, i.e., $\phi : X \to \mathbb{R}$, where $\mathbb{R}$ is the real number field. We define $\Gamma = \sup_{\phi \in H} \phi$, that is to say, $\Gamma(x) = \sup_{\phi \in H} \phi(x)$, $x \in X$. Let G be a subset of X, we consider optimization problem

$$(\Gamma, G) : \qquad \inf_{g \in G} \Gamma(g).$$

If $g_0 \in G$, it satisfies $\Gamma(g_0) = \inf_{g \in G} \Gamma(g)$, then we call g_0 an optimal solution of (Γ, G). Let $P_{(\Gamma,G)}$ be the set of all the optimal solution of (Γ, G), namely, $P_{(\Gamma,G)} = \{g_0 \in G : \quad \Gamma(g_0) = \inf_{g \in G} \Gamma(g)\}$. When ϕ is defined as $\phi_x(\cdot) = ||x - \cdot||_X$, the optimization problem (ϕ, G), in fact, is the best approximation of x by the elements of G, which has been investigated in many literatures such as [2, 3, 5, 6, 10] etc..

We have known that $\Gamma(x)$ is a convex function on Banach space X [17]. It is commonsense that convex function on X is continuous, so the function $\Gamma(x)$ is. Now we introduce some notations for convenient discussion.

$$M_{(\Gamma,x)} = \{\phi \in H : \ \ \Gamma(x) = \phi(x), \quad x \in X\}$$

$$U(x_0, \delta) = \{x : \ \ \|x - x_0\| < \delta, \quad x \in X\}$$

$$\tilde{G}_{g_0} = \bigcup_{g \in G} \{g_\alpha : \quad g_\alpha = (1-\alpha)g_0 + \alpha g, \quad \alpha \in [0,1], g_0 \in G\},$$

Definition 1. We call ϕ Fréchet differentiable at x_0, if there exists a $f_{x_0} \in X^*$ such that

$$\lim_{\|h\| \to 0^+} \frac{|\phi(x_0 + h) - \phi(x_0)) - f_{x_0}(h)|}{\|h\|} = 0$$

for all $h \in X$, and its Fréchet differential is denoted by $\partial\phi(x_0, h)$, namely, $\partial\phi(x_0, h) = f_{x_0}(h)$

Definition 2. we call Γ locally ε-maximal at x_0, if for any $\varepsilon > 0$, there exists a positive real number δ, such that for any $x \in U(x_0, \delta)$ and $\phi \in M_{(\Gamma,x_0)}$ there exists a $\varphi \in M_{(\Gamma,x)}$ such that $|\varphi(x) - \phi(x)| < \varepsilon||x - x_0||$.

Γ locally ε-maximal at x_0 implies that there exists $U(x_0, \delta)$, if $x \in U(x_0, \delta)$, we can find a $\varphi \in M_{(\Gamma,x)}$, such that

$$\lim_{||x-x_0||\to 0} \frac{|\varphi(x) - \phi(x)|}{||x - x_0||} = 0.$$

Definition 3. Assume $x_0, h \in X$, ϕ_0 is called to be Gateaux derivable at x_0 if the limit

$$\phi_0'(x_0, h) = \lim_{\alpha\to 0} \frac{\phi_0(x_0 + \alpha h) - \phi_0(x_0)}{\alpha}$$

exists for $h \in X$, and $\phi_0'(x_0, h)$ is called to be Gateaux derivative at x_0 with increment h.

It is obvious to get Lemma1 and 2.

Lemma 1. *Assume $h \in X$, and Γ be locally ε-maximal at x_0, if there exists a $\phi_0 \in M_{(\Gamma,x_0)}$ with Fréchet differential at x_0, then $\Gamma(\cdot)$ is also Fréchet differentiable at x_0 and $\partial\Gamma(x_0, h) = \partial\phi(x_0, h)$.*

Lemma 2. *Assume $h \in X$, and Γ be locally ε-maximal at x_0, if there exists a $\phi_0 \in M_{(\Gamma,x_0)}$ which is Gateaux derivable at x_0 with increment h, then $\Gamma(\cdot)$ is also Gateaux derivable at x_0 with increment h and $\Gamma'(x_0, h) = \phi_0'(x_0, h)$.*

It is commonsense that the concept of fréchet differential is stronger than that of the Gateaux, namely if $\partial\phi(x_0, h)$ exists, then $\phi'(x_0, h)$ is necessary to exist.

Theorem 3. *Assume Γ be locally ε-maximal at g_0 and there exists a $\phi_0 \in M_{(\Gamma,g_0)}$ with Fréchet differential at g_0, then $g_0 \in P_{(\Gamma,G)}$ if and only if $\partial\phi_0(g_0, g - g_0) \geq 0$ for any $g \in G$.*

Proof. Assume $g_0 \in P_{(\Gamma,G)}$, then $\Gamma(g) \geq \Gamma(g_0)$. By the assumption that Γ is locally ε-maximal at g_0 and there exists a $\phi_0 \in M_{(\Gamma,g_0)}$ with Fréchet differential at g_0 and using lemma 1, we have, for arbitrary $\varepsilon > 0$, when $\|g - g_0\| < \delta$, there exist $\delta > 0$ such that

$$\partial\phi_0(g_0, g - g_0) - \varepsilon\|g - g_0\| \leq \Gamma(g) - \Gamma(g_0) \leq \partial\phi_0(g_0, g - g_0) + \varepsilon\|g - g_0\|, \quad (1)$$

which implies $\partial\phi_0(g_0, g - g_0) \geq 0$.

On the contrary, assume that there exists a $\phi_0 \in M_{(\Gamma,g_0)}$ satisfying

$$\partial\phi_0(g_0, g - g_0) \geq 0.$$

When $\partial\phi_0(g_0, g - g_0) = 0$, we can obtain $\Gamma(g) = \Gamma(g_0)$ for all $g \in G$ by using the inequality (1). If $\partial\phi_0(g_0, g - g_0) > 0$, there exists a $\varepsilon > 0$ such that $\partial\phi_0(g_0, g - g_0) \geq \varepsilon\|g - g_0\|$, which implies that

$$\Gamma(g) - \Gamma(g_0) \geq \partial\phi_0(g_0, g - g_0) - \varepsilon\|g - g_0\| \geq 0$$

as the result of the Fréchet differentiable of ϕ_0 at g_0. □

Definition 4. We say that g_0 is a sun-point (see [14]) of Γ in G , if $g_0 \in P_{(\Gamma,G)}$ implies $g_0 \in P_{(\Gamma,\tilde{G}_{g_0})}$, where $P_{(\Gamma,\tilde{G}_{g_0})}$ is the optimal solution set of optimization problem $(\Gamma, \tilde{G}_{g_0})$. If every point $g \in G$ is a sun-point of Γ , we refer G to be a sun-set of Γ.

The notion of sun-point plays important role in nonlinear best approximation, which has been collected in a book [15].

Theorem 4. *Assume G be a sun-set of Γ and Γ locally ε-maximal at g_0 and $\phi_0 \in M_{(\Gamma,g_0)}$ with Gateaux derivative at g_0 with increase $g - g_0$, where $g \in G$, then $g_0 \in P_{(\Gamma,G)}$ if and only if $\phi_0'(g_0, g-g_0) \geq 0$ for all $g \in G$.*

Proof. Assume Γ be locally ε-maximal at g_0 and there exists a $\phi_0 \in M_{(\Gamma,g_0)}$ with Gateaux derivative at g_0 with increase $g - g_0$. By using lemma 2, we get

$$\phi_0'(g_0, g-g_0) = \lim_{t \to 0^+} \frac{\phi_0(g_0 + t(g-g_0)) - \phi_0(g_0)}{t},$$

which implies

$$\phi_0'(g_0, g-g_0) = \lim_{t \to 0^+} \frac{\Gamma(g_0 + t(g-g_0)) - \Gamma(g_0)}{t} \tag{2}$$

By the assumption that G is a sun-set of Γ and $g_0 \in P_{(\Gamma,G)}$, we have $\phi_0'(g_0, g-g_0) \geq 0$ from the equality (2). This is end of the proof of necessity of the theorem.

Now we prove the sufficiency of the theorem. Function $\gamma(t) = (\Gamma(g_0 + t(g-g_0)) - \Gamma(g_0))/t$ is an increasing function on $(0,1]$[17], consequently we get $\gamma(1) \geq \lim_{t \to 0^+} \gamma(t)$, namely

$$\Gamma(g) - \Gamma(g_0) \geq \lim_{t \to 0^+} \frac{\Gamma(g_0 + t(g-g_0)) - \Gamma(g_0)}{t} = {\phi_0}'(g_0, g-g_0) \geq 0,$$

by $\phi_0'(g_0, g-g_0) \geq 0$, where $\phi_0 \in M_{(\Gamma,g_0)}$, which implies that $g_0 \in P_{(\Gamma,G)}$. □

3 Generalized Saddle Point

Let X be a Banach space, $\tilde{H}$ be a set of all the real-valued and convex functions on X. Now we define a functional $\Psi : (\tilde{H}, X) \to R$, that is

$$\Psi(\phi, x) = \phi(x), \qquad \forall (\phi, x) \in (\tilde{H}, X).$$

Let H be a subset of $\tilde{H}$, and G be the subset of X, we also define

$$\Gamma(x) = \sup_{\phi \in H} \phi(x) = \sup_{\phi \in H} \Psi(\phi, x),$$

then the optimization problem (Γ, G) changes to be

$$(\Gamma, G): \quad \inf_{g \in G} \Gamma(g) = \inf_{g \in G} \sup_{\phi \in H} \Psi(\phi, g).$$

Definition 5. Let $(\phi_0, g_0) \in (H, G)$, we call (ϕ_0, g_0) to be generalized saddle point (GSP) of Ψ in (H, G), if it satisfies $\Psi(\phi, g_0) \leq \Psi(\phi_0, g_0) \leq \Psi(\phi_0, g), \qquad (\phi, g) \in (H, G)$.

The concept of GSP had been introduced in [17]. The notion of saddle point is a fundamental concept in many areas of science and economics. A classical instance is the famous saddle point theorem for a zero-sum matrix game due to J. Von Neumann and O. Morgenstern [1].

Theorem 5. *Assume G be a sun-set of Γ and Γ locally ε-maximal at g_0 and $\phi_0 \in M_{(\Gamma, g_0)}$ with Gateaux derivative at g_0 with increase $g - g_0$, where $g \in G$, then following statements are equivalent,*

(1) $g_0 \in P_{(\Gamma, G)}$;
(2) (ϕ_0, g_0) *is GSP of* Ψ *in* (H, G);
(3) ${\phi_0}'(\bar{g}, g - g_0) \geq 0, \quad g \in G$;
(4) $\Gamma'(g_0, g - g_0) \geq 0$.

Proof. $(2) \Rightarrow (3)$. Assume (ϕ_0, g_0) is a GSP of Ψ in (H, G), we have $\Psi(\phi, g_0) \leq \Psi(\phi_0, g_0) \leq \Psi(\phi_0, g)$, namely $\phi(g_0) \leq \phi_0(g_0) \leq \phi_0(g)$. Therefor, by the inequality and convexity of ϕ_0, we can get

$$\begin{aligned} \phi_0'(g_0, g - g_0) &= \lim_{\alpha \to 0^+} \frac{\phi_0(g_0 + \alpha(g - g_0)) - \phi_0(g_0)}{\alpha} \\ &\geq \lim_{\alpha \to 0^+} \frac{(1 - \alpha)\phi_0(g_0) + \alpha\phi_0(g) - \phi_0(g_0)}{\alpha} \\ &= \lim_{\alpha \to 0^+} \frac{\alpha[\phi_0(g) - \phi_0(g_0)]}{\alpha} \\ &= \phi_0(g) - \phi_0(g_0) \geq 0. \end{aligned}$$

$(3) \Rightarrow (2)$. Assume that $\phi_0'(g_0, g - g_0) \geq 0$. Let $\gamma(\alpha) = \frac{\phi_0(g_0 + \alpha(g - g_0)) - \phi_0(g_0)}{\alpha}$, where $\alpha \in (0, 1]$. When ϕ is a convex function, It had been proved that $\gamma(\alpha)$ is an increasing function on $(0, 1]$ [17]. So we have $\gamma(1) \geq \lim_{\alpha \to 0^+} \gamma(\alpha)$, which implies $\phi_0(g) \geq \phi_0(g_0)$. Furthermore, $\phi(g_0) \leq \sup_{\phi \in H} \phi(g_0) = \phi_0(g_0)$, because of $\phi_0 \in M_{(\Gamma, g_0)}$. So we get

$$\Psi(\phi, g_0) \leq \Psi(\phi_0, g_0) \leq \Psi(\phi_0, g), \qquad (\phi, g) \in (H, G).$$

$(1) \Leftrightarrow (3)$ and $(3) \Leftrightarrow (4)$ can be obtained by theorem 4 and Lemma 2 respectively. □

Now we can get the relationship between GSP and Fréchet differentiable in the following by means of Theorem 5.

Theorem 6. *Let $\phi_0 \in M_{(\Gamma, g_0)}$ be Fréchet differentiable at g_0, then $(\phi_0, g_0))$ is a GSP of Ψ in (H, G), if and only if $\partial\phi_0(g_0, g - g_0) \geq 0, \quad g \in G$.*

From the proofs of the above theorems, we realize that the optimal solution of (Γ, G) is equivalent to Fréchet differentiable , but not true to Gateaux derivable without G being a sunset of Γ, which displays that Fréchet differentiable is stronger than that of Gateaux derivable again.

4 Application in Best Simultaneous Approximation

Let X be a *Banach* space, X^* be its dual space, and $\|\cdot\|$, $\|\cdot\|_{X^*}$ be their norms of the space X and X^* respectively. B^* denotes the unit sphere of X^*, which is $B^* = \{f : ||f||_{X^*} = 1, \quad f \in X^*\}$. Let F be a bounded, closed and convex subset of X, and G also be a arbitrary subset of X, we define $d(F,G) = \inf_{g\in G}\sup_{x\in F}||x-g||$. We expect, in approximation theorem, to find a $g_0 \in G$ such that

$$\sup_{x\in F}||x-g_0|| = \inf_{g\in G}\sup_{x\in F}||x-g||, \tag{1}$$

then g_0 is called best simultaneous approximation of F by the elements of the set G, all of which denote $P_{(F,G)}$, namely, $P_{(F.G)} = \{g_0 : \sup_{x\in F}||x-g_0|| = d(F,G)\}$.The best approximation of a bounded set, namely, relative Chebyshev center, were discussed in many literatures [4, 7, 12, 13, 16].

Given $(f,x) \in (B^*,F)$, We define $\phi_{(f,x)}(y) = |f(x-y)|, y \in X$, where (B^*,F) is the Cartesian product set of B^* and F, which is equipped with a product topology. It is evident to get that $\phi_{(f,x)}(\cdot)$ is a nonnegative convex function on X. Let

$$H_{(B^*,F)} = \{\phi_{(f,x)} : \ \ \phi_{(f,x)}(y) = |f(x-y)|,\ (f,x) \in (B^*,F)\},$$

$$\Gamma_{(B^*,F)}(y) = \sup_{\phi_{(f,x)}\in H_{(B^*,F)}} \phi_{(f,x)}(y), \quad y \in X,$$

and

$$\Psi(\phi_{(f,x)}, y) = \phi_{(f,x)}(y) = |f(x-y)|.$$

It is obvious to get that $\Gamma_{(B^*,F)}(y)$ is a continuous function, Furthermore, we have

$$\Gamma_{(B^*,F)}(y) = \sup_{\phi_{(f,x)}\in H_{(B^*,F)}} \Psi(\phi_{(f,x)}, y) = \sup_{(f,x)\in(B^*,F)} |f(x-y)|. \tag{2}$$

We can obtain following lemma obviously.

Lemma 7. *For any $y \in X$, $\sup_{(f,x)\in(B^*,F)} |f(x-y)| = \sup_{x\in F}||x-y||$.*

In fact, by the definition of norm [8] and lemma 7, we can get following equalities

$$\begin{aligned} d(F,G) &= \inf_{g\in G}\sup_{x\in F}||x-g||_X = \inf_{g\in G}\sup_{(f,x)\in(B^*,F)}|f(x-g)| \\ &= \inf_{g\in G}\Gamma_{(B^*,F)}(g) = \sup_{\phi_{(f,x)}\in H_{(B^*,F)}} \Psi(\phi_{(f,x)}, g), \end{aligned}$$

which implies that the problem of best approximation of the set F by the elements of the set G is equivalent to the convex optimization of $(\Gamma_{(B^*,F)}, G)$.

For proving the following lemma, it is necessary to know that when X is a Banach space and M a convex subset of X, then M is a strongly closed set if and only if M is weekly closed [11].

Lemma 8. *Let F be a closed, bounded and convex subset of X, then $M_{(\Gamma_{(B^*,F)},y)}$ is not empty.*

By linearity of the function $\phi_{(f_0^*,x_0^*)}$, it is easy to have following lemma.

Lemma 9. *let $\phi_{(f_0^*,x_0^*)} \in M_{(\Gamma_{(B^*,F)},y_0)}$, then $\phi_{(f_0^*,x_0^*)}$ is Fréchet differentiable and Gateaux derivable at y_0 and $\phi'_{(f_0^*,x_0^*)}(y_0,h) = \partial\phi_{(f_0^*,x_0^*)}(y_0,h) = -f_0^*(h)$, where $h \in X$.*

Lemma 10. *[11] X is a smooth space $\Leftrightarrow$ X is Gateaux derivable space.*

Now we give following theorem about characteristic of best simultaneous approximation of F by the element of G.

Theorem 11. *Let X be smooth space, $F \subset X$ closed, bounded and convex subset of X, and G also a subset of X, $g_0 \in G$, then the following statement are equivalent,*

(1) $g_0 \in P_{(F,G)}$
(2) *there exists* $(f_0^*,x_0^*) \in M_{(\Gamma_{(B^*,F)},g_0)}$ *such that* $f_0^*(g_0 - g) \geq 0, \ \forall g \in G$
(3) *there exists* $(f_0^*,x_0^*) \in M_{(\Gamma_{(B^*,F)},g_0)}$ *such that* $(\phi_{(f_0^*,x_0^*)}, g_0)$ *be the GSP of* Ψ *in* $(H_{(B^*,F)}, G)$.

Proof. We only need prove that function $\Gamma_{(B^*,F)}(\cdot)$ is locally ε-maximal at g_0, then the proof of these equivalences are easy to get by theorem 5.

Using lemma 8, we have that there exist a $(f_0^*,x_0^*) \in M_{(\Gamma_{(B^*,F)},g_0)}$, such that

$$\Gamma_{(B^*,F)}(g_0) = f_0^*(x_0^* - g_0) = ||x_0^* - g_0|| = \sup_{x\in F} ||x - g_0||.$$

By the assumption that X is a smooth space, we can know that f_0^* is the unique supporting functional of $x_0^* - g_0$, therefore

$$\lim_{\lambda\to 0} = \frac{||x_0^* - (g_0 + \lambda h)|| - ||x_0^* - g_0||}{\lambda ||h||} = -f_0^*(h),$$

by using lemma 9 and 10, where $||h|| = 1$. This equalities implies that

$$f_\lambda^*(x_0^* - (g_0 + \lambda h) = f_0^*(x_0^* - g_0) + \lambda(-f_0^*(h)) + O(||\lambda h||), \tag{3}$$

where $f_\lambda^*, f_0^* \in B^*$ satisfying $f_\lambda^*(x_0^* - (g_0 + \lambda h)) = ||x_0^* - (g_0 + \lambda h)||$, $f_0^*(x_0^* - g_0) = ||x_0^* - g_0||$ respectively. Moreover, we have $f_0^*(x_0^* - (g_0 + \lambda h) = f_0^*(x_0^* - g_0) + \lambda(-f_0^*(h))$. And using equalities (3), hence we can get $f_\lambda^*(x_0^* - (g_0 + \lambda h) - f_0^*(x_0^* - (g_0 + \lambda h) = O(||\lambda h||)$, which implies that for any $\varepsilon > 0$, there exists $\delta > 0$ and $(f_\lambda^*, x_0^*) \in M_{(\Gamma_{(B^*,F)},(g_0+\lambda h))}$, when $||\lambda h|| < \delta$, such that

$$\frac{|f_\lambda^*(x_0^* - (g_0 + \lambda h) - f_0^*(x_0^* - g_0)|}{||\lambda h||} < \varepsilon,$$

that is to say $\Gamma_{(B^*,F)}$ is locally ε-maximal at g_0.

□

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