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Study of Water Resource Allocation Mechanism Design*

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Abstract In this paper, we design a water resource allocation mechanism which plays the role of incentives for promotion of the water use efficiency and productivity by taking water distribution for enterprices as an example. It firstly divided the water allocation problem into three cases according to the relationship among amounts of water allocatable and demanded for enterprises. By discussing the three cases respectively, we then design corresponding distribution mechanism based on incentive mechanism, combination of DEA theory and Talmud rule, and elimination mechanism to solve the problem. Numerical study in the end testifies the rationality and efficiency of the distribution mechanism in sense of guiding water-saving awareness and improvement of water efficiency.

Keywords Mechanism Design; Water Resource Allocation; Incentive Mechanism; Data Envelopment Analysis; Talmud Rule

1 Introduction

Low water efficiency and serious waste of water are universal phenomena today, which intensify conflict between supply and demand of water resources, and even impede the sustainable development of economy. Public agencies responsible for water allocation and control are calling for analytical tools of planning. Hence, a systematical distribution mechanism especially designed to solve this problem is relevant for governments. Existing resource allocation methods research focus on studying distribution scheme in accordance with uniform weight[1, 2], or allocation solution calculated by solving a non-linear optimization model with maximizing the total net benefits[3]. Neither of them, however, take into account the productivity and water consumption efficiency.

In this paper, to achieve the objective of encouraging water consumption efficiency, we design a water resource allocation mechanism by taking water distribution for enterprices as an example. In order to make right decision-making, government should first forecast the minimum water needs for survival and the highest demand for water of this year for enterprises in an industry. We utilize method and theory of data envelop analysis (DEA) [4, 5, 6, 7, 8] to evaluate the relative water use efficiencies and returns to scale

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for enterprises. The product of these two evaluation values and the highest demand for water is the efficient water demand for every firm. We then divide the problem into three cases according to relationship among allocatable water quantity and amounts of water demanded with the three kinds for enterprises. Oriented to the three cases, we present corresponding distribution methods based on incentive mechanism, combination of DEA theory and "Talmud rule" [9, 10], and elimination mechanism respectively to solve the problem. The proposed allocation mechanism not only guides water-saving awareness of the industry, but also takes the economic development of the industry into account.

2 Literature Review

2.1 CCR Model in DEA

Charnes, Cooper and Rhodes[4, 5] introduced the CCR ratio definition which generalized the single-output to single-input classical engineering-science ratio definition to multiple outputs and inputs without requiring preassigned weights. This is done via the extremal principle incorporated in the model (CCR_{FP}), which can be replaced by a linear program whose dual model (CCR_{DLP}) is commonly used in practice.

2.2 Returns to Scale in DEA

Because the real production function is difficult to calculate, Banker et al. [6] adopted the *BCC* frontier in DEA to approximate the scale economies frontier, and under this assumption, the returns to scale (RTS) concept was introduced. In VRS context, increasing, constant, decreasing returns to scale(IRTS,CRTS, DRTS), and congestion are used to measure the scale economies of different DMUs. Wei,Q.L. et al. [8] have further researched the estimation[7] of RTS by DEA theory, and proposed testing conditions for the above four categories. (x_0, y_0) denotes a point in the production possibility set, the procedure of determination for DMU_0 is following:

$Step1$: Solve P_{NEW}, and z^* be the optimum value. Let $\hat{y}_0 = z^* y_0$.

$Step2$: Solve P_{FG} and P_{ST} for $(x_0, \hat{y}_0)$.

$Step3$: Congestion and different returns to scale can be tested according to Table 1. If the DMU_0 is weakly(FG) DEA efficient, it is denoted by FG, otherwise, FG^*. Notations ST and $(ST)^*$ have a similar meaning. (P_{NEW}),(P_{FG}) and (P_{ST}) are corresponding to the testing models in DEA.

$$(P_{NEW})\ Max\quad z \qquad \text{s.t.} \sum_{j=1}^{N} x_j\lambda_j = x_0,\quad \sum_{j=1}^{N} y_j\lambda_j \geq z y_0,\quad \sum_{j=1}^{N} \lambda_j = 1,\quad \lambda_j \geq 0, j=1,\ldots,N$$

$$(P_{FG})\ Max\quad z \qquad \text{s.t.} \sum_{j=1}^{N} x_j\lambda_j \leq x_0,\quad \sum_{j=1}^{N} y_j\lambda_j \geq z\hat{y}_0,\quad \sum_{j=1}^{N} \lambda_j \leq 1,\quad \lambda_j \geq 0, j=1,\ldots,N$$

$$(P_{ST})\ Max\quad z \qquad \text{s.t.} \sum_{j=1}^{N} x_j\lambda_j \leq x_0,\quad \sum_{j=1}^{N} y_j\lambda_j \geq z\hat{y}_0,\quad \sum_{j=1}^{N} \lambda_j \geq 1,\quad \lambda_j \geq 0, j=1,\ldots,N$$

2.3 Talmud rule

"Talmud rule" [9, 10] proposed by economists *Robert Aumann* and *Maschler* in 1985 was resulted from game theory analysis about the distribution schemes of contested gar-

Table 1: Testing of Returns to Scale

Testing conditions	FG^*,ST	FG,ST	FG,ST^*	FG^*,ST^*
Result	IRTS	CRTS	DRTS	Congestion

ment problem and property dispute problem addressed in Jewish code *Talmud*. The following is a compact definition, where c_i and E respectively denotes the demand for agent i and the amount available to distribute. W.l.o.g, we assume it satisfies $c_1 \le c_2 \le \ldots \le c_N$.

Talmud rule *T. For each* $(c,E) \in \mathscr{C}^N$ *and each* $i \in N$,
1. If $\sum(c_j/2) \ge E$, *then* $T_i(c,E) \equiv min\{c_i/2, \lambda\}$, *where* λ *is chosen so that* $\sum min\{c_j/2, \lambda\} = E$.
2. If $\sum(c_j/2) \le E$, *then* $T_i(c,E) \equiv c_i - min\{c_i/2, \lambda\}$, *where* λ *is chosen so that* $\sum[c_j - min\{c_j/2, \lambda\}] = E$.

Li,X.Y. [11] applied Talmud rule in a water resource distribution problem.

3 Proposed Allocation System Design

Suppose the allocatable water resource quantity for an industry is E, which should be distributed to N enterprises. To a reasonable allocation strategy, government should first forecast water demand in this year for the industry and enterprises according to the development planning and the history of water consumption. Non negative real numbers l_i and h_i respectively denote minimum requirements of water and the highest water demand of firm i, with $l_i \le h_i$. The corresponding totals are separately $L = \sum_{i=1}^{N} l_i$ and $H = \sum_{i=1}^{N} h_i$. Without loss of generality, we assume h_i is sorted from small to large, which is the same to l_i, so $0 \le h_1 \le h_2 \le \ldots \le h_N$, and $0 \le l_1 \le l_2 \le \ldots \le l_N$. x_i denotes the ultimate allocation solution of water resource for enterprise i, where $x = (x_1, x_2, \ldots, x_N)$.

In this paper, we focus our discussion on case $E \in (0,H]$. When $E \in (L,H]$, to achieve the target of enhancing water-saving awareness and water efficiency, it is not reasonable to design a unified approach to allocate water resource, but requires to introduce the concept of "efficient water demand" to divide the condition of water quantity into two cases, and design different schemes for the cases respectively. By building and then solving model (CCR_{DLP}) for every firm, the optimal objective value θ_i^* as the value of relative water efficiency for firm i is obtained. From the point of view with the overall development of industry, we should also consider the production efficiency of firms. By evaluating returns to scale of firm i, the corresponding weight ρ_i for the category of returns to scale, regarded as the weight of production efficiency is stipulated by government. The weights of categories IRST, CRST, DRST and Congestion should be in non-increasing order, and satisfy $\theta_i^* \rho_i \le 1$. Then the efficient water demand for firm i is $c_i = \theta_i^* \rho_i h_i$ and the total value is $C = \sum_{i=1}^{N} c_i$. If some special case appears, like $c_i < l_i$ for some i, then fix $c_i = l_i$.

Thus, we divide the water resource allocation problem into following three cases according to the size relationship among allocatable water quantity E and amounts of water demands with three kinds L, C, H for the whole industry: case 1, $C < E \le H$; case 2, $L < E \le C$; case 3, $E \le L$.

Especially, for case 2 $E \in (L,C]$, let $D = L+(C-L)t$, where $t \in [0,1]$, then we divide interval $(L,C]$ into two scalable sub-intervals: $(L,D]$ and $(D,C]$. Case 2.1 and case 2.2 denote conditions of $E \in (D,C]$ and $E \in (L,D]$ respectively.

According to the above discussion, the system framework diagram for water resource allocation is shown in Figure 1:

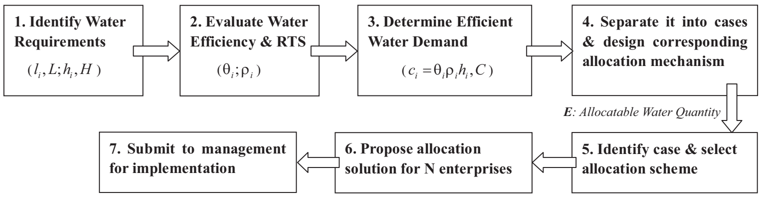

Figure 1: Flow chart for proposed allocation system design

The most essential part of the system planning is to design corresponding allocation mechanism for cases which is the step 4 shown in figure 1. Figure2 below illustrates a simple structure of cases separation and methods for designing corresponding distribution mechanism.

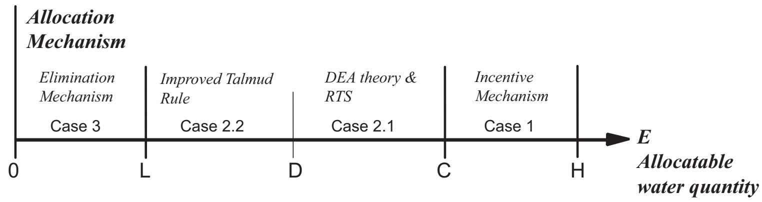

Figure 2: Framework diagram for cases and corresponding allocation mechanism

Following subsections will focus on the detailed discussion of reasonable distribution schemes for the cases respectively.

3.1 Allocation Scheme for Case 1

When $C \le E \le H$, we design a distribution scheme which plays the role of incentives for promotion of the water use efficiency and productivity of enterprises.

Algorithm 1 (Allocation Algorithm for Case1):

Let $\hat{h}_i = h_i - c_i$, $\hat{H} = \sum_{i=1}^{N} \hat{h}_i$, $\hat{E} = E - C$. Rank the value of $\rho_i \theta_i^*$ for the N firms from large to small. W.l.o.g, assume the ranking result is $k = (k_1, k_2, \dots, k_N)$,where k_i represents the firm ranked at i_{th} place in the sorting of $\rho_i \theta_i^*$ from large to small.

Step1: for i from 1 to N, compute $x_i = c_i$.

Step2: with the growth of water quantity $\hat{E}$ from 0 to $\hat{H}$, allocate water with priority order to meet firm with high efficiency, which means that for i from 1 to N, compute $\hat{x}_{k_i} = \hat{h}_{k_i} - min\{\hat{h}_{k_i}, \lambda\}$, where λ is chosen so that $\sum_{j=1}^{i} [\hat{h}_{k_i} - min\{\hat{h}_{k_i}, \lambda\}] = \hat{E}$.

Step3: for i from 1 to N, compute $x_i = \hat{x}_i + x_i$.

Output $x = (x_1, x_2, \ldots, x_N)$.

3.2 Allocation Scheme for Case 2

For case 2.1 when allocatable water resource is relative sufficient, we consider water use efficiency while allocating water resource. For case 2.2 when it is relative insufficient, proration means more difficult for development of small and medium-sized enterprises, and even bankruptcy potential, which would adversely affect economy of the whole region and industry. Thus an improved allocation scheme based on "Talmud rule" is designed and applied for case 2.2 to both ensure fairness and moderately protect the vulnerable groups. Government can balance between the two preferences of efficiency and sustainability of improvement by stipulating an appropriate value of t.

3.2.1 Allocation Scheme for Case 2.1

When $D < E \leq C$, we both consider effect of water efficiency and returns to scale of enterprises on allocation process.

If the DEA model we built is with multiple input and multiple output form, which are the vast majority of cases in practice, then it needs further discussion with input scales of firms when pertaining to allocation. From CCR ratio model, different inputs and outputs weights are allowed for evaluating different DMUs. We use following $(\theta - CWA)$ model[1] to get a set of common weights of inputs and outputs for all DMUs.

$$
\begin{aligned}
&(\theta - CWA)\ Min \quad \sum_{i=1}^{N}(|\triangle_I^i| + |\triangle_O^i|) \\
&\text{s.t.} \quad \frac{\sum_{j=1}^{s} u_j y_{ij} + \triangle_O^i}{\sum_{j=1}^{m} v_j x_{ij} + \triangle_I^i} = \theta_i, \\
&\quad i = 1,2,\ldots,N \\
&\quad u_i > 0, \quad i = 1,2,\ldots,s \\
&\quad v_i > 0, \quad i = 1,2,\ldots,m \\
&\quad \triangle_O^i, \triangle_I^i \text{ free}
\end{aligned}
$$

$$
\begin{aligned}
&(\theta - CWA_{LP})\ Min \quad \sum_{i=1}^{N}(\triangle_I^{i2} + \triangle_O^{i2}) \\
&\text{s.t.} \quad \sum_{j=1}^{s} u_j y_{ij} + \triangle_O^i - \theta_i \sum_{j=1}^{m} v_j x_{ij} - \theta_i \triangle_I^i = 0, \\
&\quad i = 1,2,\ldots,N \\
&\quad u_i > 0, \quad i = 1,2,\ldots,s \\
&\quad v_i > 0, \quad i = 1,2,\ldots,m \\
&\quad \triangle_O^i, \triangle_I^i \text{ free}
\end{aligned}
$$

We present following allocation algorithm for case 2.1.

Algorithm 2 (Allocation Algorithm for Case2.1):

Given N enterprises, the inputs X and outputs Y, θ^*, ρ, the most compromise common weight of input and output $cw = (v_1, v_2, \ldots, v_m, u_1, u_2, \ldots, u_s)$ computed by $(\theta - CWA_{LP})$ which is equivalent to the left model $(\theta - CWA)$, it follows:

Step1: if DEA models adopted are with single input or single output form, then for i from 1 to N, compute $w_i = \frac{\rho_i \theta_i^*}{\sum_{i=1}^{N} \rho_i \theta_i^*}$, and go to step 4; else go to step2.

Step2: if models used are with multi-input and multi-output form, then for j from 1 to N, compute $\mu_j = \sum_{i=1}^{s} u_i y_{ij}$, and go to step 3.

Step3: for i from 1 to N, compute $w_i = \frac{\rho_i \mu_i}{\sum_{j=1}^{N} \rho_j \mu_j}$, and go to step 4.

Step4: for i from 1 to N, compute $x_i = w_i E$.

Output $x = (x_1, x_2, \ldots, x_N)$.

3.2.2 Allocation Scheme for Case 2.2

When $L \leq E \leq D$, which represents that the water is little, it is more difficult for the development of small and medium-sized firms. Hence, we present the following allocation algorithm based on improved "Talmud rule" for case 2.2, to both ensure fairness and the sustainability of improvement for the industry.

Algorithm 3 (Allocation Algorithm for Case2.2):

Let $\hat{E} = E - L$, and $\hat{d_i} = d_i - l_i$, for $i = 1,2,\ldots,N$, so $\hat{D} = D - L$. w.l.o.g, assume $\hat{d_1} \leq \hat{d_2} \leq \ldots \leq \hat{d_N}$.

Step1: for i from 1 to N, compute $x_i = l_i$.

Step2: let $\hat{d_0} = 0$, and define a vector w with element $w(i)$ for $i = 0,1,2,\ldots,N$. If $0 < \hat{E} < \hat{D}/2$, then compute $w(0) = 0$, and for i from 1 to N, compute $w(i) = \sum_{j=0}^{i-1} \hat{d_j}/2 + \hat{d_i} \times (N+1-i)/2$, and go to Step3; else, for i from 1 to N, compute $w(i) = \sum_{j=0}^{N-i} \hat{d_j}/2 + \sum_{j=n-i+1}^{N} (\hat{d_j} - \hat{d}_{N-i}/2)$, and go to Step4.

Step3: search j from 1 to N, until it satisfies $w(j-1) < \hat{E} \leq w(j)$. Then for k from 1 to $j-1$, compute $\hat{x}_k = \frac{\hat{d_k}}{2}$; and for k from j to $k = N$, compute $\hat{x}_k = (\hat{E} - \sum_{s=1}^{j-1} \hat{x}_s)/(N+1-j)$.

Step4: search j from 1 to N, until it satisfies $w(j-1) < \hat{E} \leq w(j)$. Then for k from 1 to $N-j$, compute $\hat{x}_k = \hat{d}_k/2$; and for k from $N-j+1$ to N, compute $\hat{x}_k = (\hat{E} - \sum_{s=1}^{N-s} \hat{x}_s)/j$.

Step5: for i from 1 to N, compute $x_i = x_i + \hat{x}_i$.

Output $x = (x_i, x_2, \ldots, x_N)$.

The algorithm 4 solve the problem in case 2.2 by transforming it to a two steps allocation. Government first distribute l_i to firm i to meet the survival demand, and then allocate the rest quantity of water $\hat{E} = E - L$ to the N enterprises by the improved talmud rule. The final solution is the sum of the two step allocation results.

3.2.3 Extreme Cases

The above two allocation schemes for case2.1 and case2.2 hold in general. However, some special situations may happen as follows: when it comes to $L < D < E \leq C$ or $L < E \leq D$, there are $x_{i_j} < l_{i_j}, i_j = 1,2,\ldots,k$ for k firms. To solve the special case, set $x_{i_j} = l_{i_j}$ for all $i_j = 1,\ldots,k$ and adopt corresponding allocation schemes for the rest $N-k$ firms respectively to finish the distribution process of the rest quantity of water $E - \sum_{i_j=1}^{k} l_{i_j}$.

3.3 Allocation Scheme for Case3

For case3, when enterprises are all facing the risk of bankruptcy. We discuss an elimination scheme with selection of several firms to be closed to meet the basic water demands for the rest firms, maintaining a more effective development of the whole industry. In this paper we consider sorting $\theta^*\rho$ for the N firms from large to small. The smaller the value of firm, the sooner the firm to be eliminated.

4 Numerical Study

In this section, an allocation procedure for three cases is illustrated with numerical example. Suppose the water demands of this year for the N enterprises in an industry are forecasted by government in Table 2 below, and the total of highest demanding is 129.5, that of survival request of water is 55.5. We discuss three cases of $E = 110$, $E = 75$, $E = 50$ successively. (Unit: 10,000 tons, the same for Table 2)

Table 2: Demands for Water Resource

	A	B	C	D	E	F	G	H	I	J	K	L
l_i	10	8	7	6	5	4	4	3	2.5	2	2	2
h_i	30	20	15	12	10	8	7.5	7	6	5	5	4

Table 3 mentions the datum for the 12 firms: input for annual water consumptions and annual waste water emissions; output for annual production and annual production value. (Unit: 10M RMB for output2, 10,000tons for others)

Table 3: Input and Output Data of Example

DMU	A	B	C	D	E	F	G	H	I	J	K	L
Input1	28	18	14	11	9	8	6.5	6	5.5	4.5	4.5	3
Input2	12	10	9	4	3.5	3	2.8	2.2	1.8	1.2	1	0.5
Output1	32	20	15	11	12	10	8	6	2.5	2.5	2	1
Output2	80	65	35	28	25	20	17	14	6	6	4	3

Building and then solving DEA models with above datum, we can get relative water efficiencies: $\theta = (0.96, 1, 0.84, 0.98, 1, 0.97, 0.93, 0.89, 0.62, 0.7, 0.58, 0.7)$, which is followed by the test of RTS for the 12 firms. Except for firms G and L with IRTS, firms B and E with CRTS, firm A with DRTS, other firms all belong to congestion. Assume that the weights stipulated by government for (IRTS),(CRTS),(DRTS), and congestion are: 1.075,1,0.9,0.8. Thus the total efficient demand of water resource is $C = 105.206$.

(1) $E = 110$ satisfies Case1. The order of the twelve firms with the value of $\rho\theta^*$ from large to small is B,E,G,A,D,F,L,H,C,J,I,K. We can adopt distribution scheme based on incentive mechanism to obtain the final allocation solution.

(2) $E = 75$ satisfies $L \le E \le C$. If government stipulates $t = 0.3$, then $D = 70.4118$, and the problem belongs to case2.1. For the model utilized is multi-inputs and multi-outputs form, we figure out the compromise common weights: $cw = (u_1, u_2, v_1, v_2) = (0.0695, 0.2025, 0.2906, 0.3384)$, combined with the weight of returns to scale ρ to get the final allocation result. If $t = 0.7$, then $D = 90.2942$, and it belongs to case2.2 $E \in (L, D]$. By Algorithm 4, allocation based on improved "Talmud rule", we obtain the final result $x = \hat{x}_i + l_i$.

(3)When $E = 50$, which satisfies $E \le L$, by sorting the values $\theta^*\rho$, we finally decide to close firms I,K, and allocate E for the rest 10 firms.

The allocation results for all of the cases are shown in Table 4 below:

Table 4: Allocation Results of Example

	A	B	C	D	E	F	G	H	I	J	K	L
1	30	20	10.1	10.1	10	6.21	7.5	7	6	5	5	4
2.1	19.7	15.6	7.96	6.07	7.58	4.97	5.5	3.18	1.34	1.34	1	0.8
2.2	17.7	12.2	8	7.2	6.75	4.77	5.2	3.7	2.67	2.28	2.1	2.4
3	10	8	7	6	5	4	4	3	0	1	0	2

5 Conclusion

In this paper, we proposed a water resource distribution mechanism with encouragement of improving water use efficiency and guidance of water-saving awareness for enterprises. This mechanism can be not only used to solve the water resource distribution but also applicable to other kinds of resource allocation or limited supplies distribution problem. We preliminarily studied the designing of elimination mechanism for case 3, which will be studied for further detailed discussion in the future.

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