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Model of Integrated Chance Constraints and Application on Risk Measure

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Abstract Compared with chance constraints model, integrated chance constraints model has better property about feasible solution set and measures risk more accurately in economy control. In this paper, we introduce four definitions of ICC models, and discuss the properties including convexity, continuum and differentiability. Then we present a hybrid intelligent algorithm, which is test efficiently by numerical experiments. At last, as a new methord to measure risk, conditional valu-at risk is studied as the application of ICC. Mean-CVaR problem can be computed efficiently using our model and algorithm, which dramatically improves the portfolios of investment.

Keywords chance constraints programming; intelligent algorithm; CVaR

1 Introduction

As an important model of stochastic programming, chance constraints programming has been used successfully in a wide range of applications [1-2]. However, It is well known that chance constrained problems are non-convex in general, and they are convex only if certain rather strong conditions on the distribution of the underlying random vector is satisfied. As a quantitative alternative for traditional chance constraints, integrated chance constraints model was proposed by R.J.B.Wets [3].

In Section 2 of this paper we introduce models of ICC. Then we discuss properties of ICC with stochastic convex constraints in Section 3, and get the properties including convexity of solution set, continuum and differentiability of restriction function. Subsequently, based on these properties, we design a hybrid intelligent algorithm for solving ICC model with discrete random variable in Section 4. Last in Section 5 we apply ICC model on conditional valu-at risk (CVaR) to measure risk.

2 Models of ICC

We consider stochastic linear programming that random variable exists only in

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constraint. $\min_{x\in D} \{c^T x | A(\omega)x \ge b(\omega)\}$, where $A(\omega)$ and $b(\omega)$ are random matrix vector seperately. Denoting $\eta_i(x,\omega) = \sum_{j=1}^{n} a_{ij}(\omega)x_j - b_i(\omega), i = 1,2,\cdots,m$, then $\eta_i^- = \max\{0, -\eta_i(x,\omega)\}$ can be thought as the shortage of resource. When η_i^- is continues variable, the average shortage is $E\eta_i^- = \int_{-\infty}^{0} \Pr(\eta_i \le t)dt$. According to [1,3], the first definition of ICC is as follows.

Definition 2.1.

The individual model of ICC is defined as $E\eta_i^- \le \beta_i, i = 1,2,\cdots,m$. Where β_i is maximum acceptance value given in advance, and the feasible set is: $X_i^1(\beta_i) = \{x \in R^n, E\eta_i^- \le \beta_i\}$. The joint ICC is $E(\eta_i^-, i = 1\cdots m) \le \beta$, and the feasible set is $X(\beta) = \cap_{i=1}^{m} X^1{}_i(\beta_i)$.

Definition 2.2.

The joint ICC is $E[\max_i(\eta_i^-)] \le \beta$, $i = 1,2,\cdots,m$, and the feasible set is $X_i^2(\alpha_i) = \{x \in R^n, E[\max_i(\eta_i^-)] \le \beta\}$.

The surplus and shortage are considered together, where $E\eta_i^- + E\eta_i^+ = E|\eta_i|$.

Definition 2.3.

The individual ICC has the form $E\eta_i^- \le \alpha_i E|\eta_i|, i = 1,2,\cdots,m$, where α_i is the maximum acceptance value given in advance, and the feasible set is $X_i^3(\alpha_i) = \{x \in R^n, E\eta_i^- \le \alpha_i E|\eta_i|\}$. The joint ICC is $E(\eta_i^-, i = 1\cdots m) \le \alpha$ and the feasible set is $X(\alpha) = \cap_{i=1}^{m} X_i^3(\alpha_i)$.

$\gamma \in [0,\infty)$ is introduced as the maximum acceptance value of condition expectation [4], so

Definition 2.4.

The ICC is $E\eta_i^- \le \gamma \cdot P(\eta_i < 0)$, and the feasible set is $X^4(\gamma) = \{x \in R^n, E\eta_i^- \le \gamma \cdot E|\eta_i|\}$.

Remark For $P(\eta_i \le 0) = E[\operatorname{sgn}(\eta_i)^-]$, so the chance constraint is a special ICC.

3 Properties of ICC

The reference [3] gave some properties of the ICC model with stochastic linear constraints, which includes the convexity of solution set, continuum and differentiability of restriction function. In this section, we discuss ICC model with stochastic convex constraints.

For $x \in R^n$ and $\omega \in S^p$, let $A(x,\omega) \subseteq S^p$ is the ω-area satisfying $g(x,\omega) \le 0$, then we have $g(x,\omega)^- = g(x,\omega) \times \mathrm{I}_A(x,\omega)$, where $\mathrm{I}_A(x,\omega) = \begin{cases} 1, & \omega \in \mathrm{A}(x,\omega) \\ 0, & \omega \notin \mathrm{A}(x,\omega) \end{cases}$.

Lemma 3.1.

$E(g(x,\omega) \times \mathrm{I}_B(x,\omega)) \le E(g(x,\omega) \times \mathrm{I}_A(x,\omega)) = E(g(x,\omega)^-)$ holds for any $B \ne A$ in S^p.

Theorem 3.2.

For stochastic optimazation problem $\min\{ f(x) : g(x,\omega) \ge 0, \mathrm{x} \in \mathrm{D}\}$, where D is a fixed, closed, finite set, $g(x,\omega)$ is convex about x and each random variable satisfying $E(\omega_j) < \infty$, the following conclusions hold.

(1) $\bar{g}(x) = E[g(x,\omega)^-]$ is finite, nonnegative, convex and Lipschits continuous.

(2) For finite discrete distribution of $(a(\omega), b(\omega))$, $\bar{g}(x)$ is piece-wisely convex function and is continuous and differentiable for continuous distribution of $(a(\omega), b(\omega))$. So the set $X(\beta) = \{x \in R^n : \bar{g}(x) \le \beta\}$ is convex.

(3) For $g(x,\omega) = \sum_{j=1}^{n} a_j(\omega) x_j^2 - b(\omega)$, it satisfies $\frac{\partial \bar{g}(x)}{\partial x_j} = E[-2a_j x_j \operatorname{sgn}(g(x)^-)]$, moreover, $\lim_{\lambda \to \infty} \frac{\bar{g}(x+\lambda y) - \bar{g}(x)}{\lambda^2} = E[(\sum_{j=1}^{n} a_j(\omega) y_j^2)^-]$.

Proof Firstly prove $\bar{g}(x)$ is convex. For any $\lambda \in [0,1]$, $x^1, x^2 \in R^m$, $x^0 = \lambda x^1 + (1-\lambda) x^2$, then

$$
\begin{aligned}
\bar{g}(x^0) &= E\{g(\lambda x^1 + (1-\lambda)x^2, \omega) \times \mathrm{I}_A(x^0,\omega)\} \\
&\le E\{[\lambda g(x^1,\omega) + (1-\lambda) g(x^2,\omega)] \times \mathrm{I}_A(x^0,\omega)\} \\
&= \lambda E\{g(x^1,\omega) \times \mathrm{I}_A(x^0,\omega)\} + (1-\lambda) E\{g(x^2,\omega) \times \mathrm{I}_A(x^0,\omega)\} \\
&\le \lambda E[g(x^1,\omega)^-] + (1-\lambda) E[g(x^2,\omega)^-] = \lambda \bar{g}(x^1) + (1-\lambda) \bar{g}(x^2).
\end{aligned}
$$

Basing on the definition of convexity, we know that $\bar{g}(x)$ is convex about x, moreover is Lipschits for any x lying in D. So $X(\beta) := \{x \in R^n : \bar{g}(x) \le \beta\}$ is convex for given β.

Secondly, according to the supposing that $\overline{g}(x)$ is bounded and negative.

Especially, when $g(x,\omega)=\sum_{j=1}^{n} a_j(\omega)x_j^2 - b(\omega)$, $\overline{g}(x)=E[g(x,\omega)^-]$ hold and random variable has continuous distribution, we have following results.

For any pair (p, q) of real number satisfying p=0 or $|p| \geq |q|$, the equation holds that $(p+q)^- = p^- + q^- - \{q^- \operatorname{sgn} p^+ + q^+ \operatorname{sgn} p^-\}$.

Denoting $p=[\sum_{j=1}^{n} a_j(\omega)(x_j+\lambda y_j)^2 - \sum_{j=1}^{n} a_j(\omega)x_j^2]$ and $q=\sum_{j=1}^{n} a_j(\omega)x_j^2 - b(\omega)$, then for any fixed $(a,b)\in R^{n+1}$, $\lim_{\lambda\to\infty}\frac{(p+q)^- - q^-}{\lambda^2} = E[(\sum_{j=1}^{n} a_j(\omega)y_j^2)^-]$ holds.

Because $\max_i f_i(x)$ is polyhedron function, the theorem 3.3 holds for ICC model.

Theorem 3.3.

The set $Y(\beta):=\{x\in R^n, E[\max_{\mathrm{i}}(\eta_i^-)]\leq\beta\}$ is convex for $0\leq\beta<+\infty$.

4 Hybrid Intelligent Algorithm for ICC with Discrete Random Variable

In this section, we study the algorithm for ICC model with discrete random variable of which the distribution function is known. In general the programming with stochastic constraints is as

$$\min_{x\in D} cx$$
$$s.t.\ \ T(\omega)x \geq h(\omega)$$

where *x* and *c* are n dimension vector, $D=\{x\in R_+^n : Ax=b\}$ is a set of *x* which is negative and equivalence, ω is sample point in probability space (Ω, F, P), $T(\omega), h(\omega)$ are random vectors and $T_i(\omega)x\geq h_i(\omega)$ holds for all $i=1,2,\cdots m$.

Let S is the observation set, and ω^k means the k^{th} sample of ω , $k\in S=\{1,2,\cdots|S|\}$, and $\eta_i(x,\omega)=T_i(\omega)x-h_i(\omega)$. So the expection of stochastic constraint can be written as $E[\eta_i^-]=\sum_{k=1}^{|S|} p^k \times\max\{0,-\eta_i(x,\omega^k)\}$, where $\Pr\{\omega=\omega^k\}=p^k$.

When random variable obeys the discrete distribution, the most difficulty of computing ICC is focused on the computing of the $E[\eta_i^-] \le \beta$. Especially when the numbers of n and |S| are very large, the classical algorithm such as simplex method and test software are not suitable any more. Basing on the study in Section 2 and 3, we design the hybrid intelligent algorithm according to the character of the model, and assuming all risk parameters of β_i have the same value of β in this section. So we have relevant theorem as follows.

Theorem 4.1.

Leting $\Pr\{\omega = \omega^k\} = p^k, k \in S$, $\beta \ge 0$, then $E[\eta_i^-] \le \beta$ has inequality form as follows

$$\begin{cases} T_i^k + y_i^k \ge h_i^k, k \in S \\ \sum_{k \in S} p^k y_i^k \le \beta \\ y_i^k \ge 0, \quad k \in S \end{cases}$$

Proof

$$E_\omega[\eta_i^-] = \sum_{k \in S} p^k \max\{0, -\eta_i(x, \omega^k)\}$$

$$= \sum_{k \in S} \max\{0, -p^k \eta_i(x, \omega^k)\} = \sum_{k \in S} p^k [-\eta_i(x, \omega^k)]^+ .$$

For $y_i^k = [-\eta_i(x, \omega^k)]^+$, $y_i^k \ge 0$ holds which satisfis the given condition.

So the feasible set is $X(\beta) = \cap_{i=1}^m X_i(\beta)$, where $X_i(\beta) = \{x \in R^n, E[\eta_i^-] \le \beta\}$. $X_i(\beta)$ is a polyhedron composed by the primal n variables, new |S| variables and |S|+1 linear constraint. Denoting $\{T_i^k, h_i^k\} = \{T_i(\omega^k), h_i(\omega^k)\}$, we have

Theorem 4.2

$E[\eta_i^-] = \max_{K \subseteq S}\{\sum_{k \in K} - p^k \eta_i(x, \omega^k)\}$ holds for all $\beta \ge 0$.

From theorem 4.2, we know that when S is finite set the corresponding set to each constraint has the form as $X_i(\beta) = \bigcap_{K \subseteq S} \{x \in R^n : \sum_{k \in K} p^k (h_i^k - T_i^k x) \le \beta \}$. The $X_i(\beta)$ is a polyhedron composed by the primal n variable and the new $2^{|S|} - 1$ linear constraints. Subsequently, we make use of the character of ICC by appending ceaselessly the subset K that has the most number from S, and combine genetic, climb algorithm to design the hybrid intelligent algorithm. It is composed of genetic algorithm which is good at global searching and hill climbing algorithm which is fitable in local searching.

Algorithm4.3.

Step1 Generate randomly the first population with N individuals in the set

$D \cap B_i^0$ and t:=1. Where $B_i^0 = \{x \in R^n : \overline{T}_i x \geq \overline{h}_i - \beta\}, i = 1,2,\cdots m$, $\overline{T}_i, \overline{h}_i$ are the expectation of $T_i(\omega), h_i(\omega)$ separately, set up the counter of the number of constraint that k=1.

Step2 If individuals $E[\eta_i(pop_j^t)^-] \leq \beta$ hold, compute $value(pop_j(t))$ and denote the best individual is $x^t = \arg\max_j \{value(pop_j(t))\}$, $j = 1,2,\cdots N'$.

Step3 Let k=k+1 and access new constraint $\sum_{l=1}^{|S|-k} p^l (h_i^l - T_i^l x) \leq \beta$.

Step4 Make crossover operator. If the constraint in step3 is not satisfied, take crossover again.

Step5 Make mutation operator. If the constraint in step3 is not satisfied, take mutation again.

Step6 Compute the fitness of the individual.

Step7 Select in term of roulette and get the new population.

Step8 If $| f(x^t) - f(x^{t-1})| \leq 10^{-4}$ holds, output the current solution, otherwise go to Step 2.

According to the our algorithm, two numerical experiments in [5] are test by MATLAB in the computer with PIII800, 256MEMS memory. We choose the population size is 30， the crossover probability is 0.6 and the mutation probability is 0.1.

Example 4.4 Consider the ICC model in which the object is to minimize c*x*, the determined constraint is D= $[0, 100]^2$ and the constraint is $E[(\omega_1 x_1 + \omega_2 x_2 - h)^-] \leq \beta$, where c=(-1, -2) and $\beta = 9$. The 1000 realizations of $(T(\omega), h(\omega)) = (\omega_1, \omega_2, h)$ are sampled from the uniform distribution on $[-0.5, 0.5]^2 \times [0,1]$.

In [5], the optimal solution is $\hat{x}$ = (31.71, 65.58) and $f(\hat{x})$ = -162.87. While by algorithm 4.3, the solution x^*= (71.42, 74.11) with $E[(\omega_1 x_1 + \omega_2 x_2 - h)^-] = 8.99$ are got after 98 iteration, which shows x^* is very near the borderline of feasible set. The objective function $f(x^*) = cx^*$= -219.65 is much better than that in [5] and the time cost is only 54 second.

Example 4.5 (problem generated randomly)

-- A cost vector c is sampled from the uniform distribution on $[-1, 0]^n$, thus c*x* will be minimized.

-- The set D equals to $[0, 10000]^n$.

--Assume there is one random constraint, d% of the components of the n-vector $T(\omega)$ is random, the remaining (100-d)% are deterministic entries. The fixed elements are obtained as a sample from the uniform distribution on [-1,1], whereas random elements of realizations are sampled from the uniform distribution on [-1,0] or [0,1] (d%/2 each way).

-- Realization h^k, $k \in S$ are sampled from the uniform distribution on [0,1].

-- The probabilities p^k, $k \in S$ are also drawn from the uniform distribution on [0,1] (and then normalized).

-- The risk parameter β takes value of 12.345 for all instances.

Table 1: Results according to our algorithm

S \ n	|S|=10			|S|=100			|S|=1000			|S|=5000			|S|=10000		
2	13	5	12.30	25	7	12.33	21	5	12.33	25	16	12.31	18	16	12.24
10	17	6.5	12.33	43	13	12.33	40	18	12.34	16	15	12.20	18	25	12.26
20	38	15	12.34	9	2	12.27	21	7	12.28	30	29	12.32	32	57	12.32
50	10	1	12.24	10	1	12.25	23	8.5	12.31	19	21	12.27	8	11	12.14
100	25	11	12.26	35	16	12.29	44	49	12.34	10	18	12.11	8	27	12.26
200	18	15	12.22	16	14	12.20	21	29	12.14	14	40	12.22	15	83	12.30

Remark

a) Each pari of (n, S) corresponds to a tri-group. Where the first data is the iteration number, the second is the cost time of seconds, and the third is the value E_ω of satisfying solution from which the distance betwwen the satisfying solution to the boundary of the feasible set.

b) For each (n, S), the algorithm runs three times and the average results are given in Tab.1.

c) For thr reason that some coefficient are generated randomly, the satisfying solutions can't be compared. Thus the satisfying solution isn't list in the Tab.1.

It is known from the Tab.1 that even for large-scale (n, S), the satisfying solution can be found quickly by using our algorithm. But compared with small-scale problem about (n, S), the precision decreases and the satisfying solution is near the boundary of the feasible set.

5 Application on Portfolios of Investment

Let $f(x, y)$ be a loss function depending upon the decision vector x and a random vector y. The decision vector x belongs to a feasible set of portfolios X, satisfying imposed requirements. For example, we may consider portfolios with non-negative positions. For convenience, we assume that the random vector y has aprobability density function $P(y)$. Denote by $\Psi(x,\alpha) = \int_{f(x,y)\le\alpha} p(y)dy$ the probability that the loss does not exceed threshold value $\alpha \in R$. The value-at-risk (VaR) function $VaR_\beta(x) = \alpha_\beta(x) = \min\{\alpha \in R : \Psi(x,\alpha) \ge \beta\}$ is the percentile of the loss distribution with confidence level β [6-8], and is also the smallest number such that $\Psi(x, \alpha(x, \beta)) = \beta$. An alternative measure of loss, with more attractive properties, is conditional value-at-risk, which is also called Mean-Excess loss, or Tail-VaR (under the condition that it exceeds VaR). CVaR is a more consistent measure of risk since it is sub-additive and convex[9].

$$CVaR_{\beta}(x) = \phi_{\beta}(x) = E\left[f(x,y) \mid f(x,y) \geq VaR_{\beta}(x) \right] = (1-\beta)^{-1} \int_{f(x,y) \geq VaR_{\beta}(x)} f(x,y)P(y)dy$$

VaR and CVaR satisfy[10] $\alpha_{\beta}(x) \in \arg\min_{\alpha \in R} F_{\beta}(x,\alpha)$, and $\phi_{\beta}(x) = F_{\beta}(x, \alpha_{\beta}(x))$, where $F_{\beta}(x,\alpha) = \alpha + (1-\beta)^{-1} \int (L(x,y) - \alpha)^{+} P(y)dy$.

To compute the Mean-CVaR investment problem, in which only a special case is studied about the profit obeying Gaussian distribution[11], we select four stocks (601628, 601398, 601600, 600019) from the internet of http://finance.sina.com.cn, and the data is the closing price of each day from November 14, 2008 to February 26, 2009. The value of VaR and CVaR about each stock are computed in table 2 and table 3. Both tables show that the 601398 is the smallest while the 601628 is largest as risk value is considered.

Table 2: VaR and CvaR about earnings

	VaR	CVaR
601628	5.41	6.91
601398	0.65	0.72
601600	2.65	3.82
600019	1.31	1.65

Table 3: VaR and CvaR about yield rate

	VaR	CVaR
601628	27.743%	29.655%
601398	17.361%	17.810%
601600	23.373%	30.703%
600019	21.828%	27.435%

From the analysis above, it is obvious that CVaR is a class of ICC model. Then we introduce a smooth convex Mean-CVaR portfolio model as follows.

$$\begin{aligned} & \max \ p^{T}x \\ & s.t. \ \sum_{j=1}^{4} x_j = 1 \\ & \quad CVaR_{\beta}(x) \leq c \\ & \quad x_j \geq 0, \ j = 1,2,3,4 \end{aligned}$$

As the example above is considered, the random vector is with finite number of scenarios.. Applying the algorithm 4.3, with c = 8 and $\alpha = 98\%$, the portfolio solution is (1, 0, 0, 0) and the optimal objective is 20.9776, which is consistent with the real earnings of four stocks.

6 Conclusion

This article has outlined a new model of integrated chance constraints. We have discussed the charaters, intelligent algorithm and application on CVaR measure. Our conclusion of convexity expands the linearity, numerical experiments show that the algorithm can efficiently solve ICC model with large scale. Also, as a special ICC, CVaR can be handled efficiently by algorithm 4.3, and Mean-CVaR optimal portfolios dramatically improves the method in [12], which is helpful to the decision making. There is room for much improvement and refinement of the considered approach. Additional research needs to be conducted on various theoretical and numerical aspects of the methodology.

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