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Sensitivity and Approximation of $M/G/c$ Queue: Numerical Experiments*

Yang Woo Shin[1] Dug Hee Moon[2]

[1]Department of Statistics, Changwon National University, Changwon, Gyeongnam 641-773, KOREA, e-mail: ywshin@changwon.ac.kr
[2]Department of Industrial and Systems Engineering, Changwon National University, Changwon, Gyeongnam 641-773, KOREA, e-mail : dhmoon@changwon.ac.kr

Abstract The sensitivity of the performance measures such as the mean and the standard deviation of the queue length and the blocking probability with respect to the moments of the service time are numerically investigated. The steady state distribution of $M/G/c$ queue is approximated by that of the $M/PH/c$ queue where the phase type (PH) distribution is fitted by matching the first three moments of the service times. Approximations are compared with the simulations.

Keywords $M/G/c$ queue, distribution of phase type, approximation, simulation

1 Introduction

Consider the $M/G/c$ queue with Poisson arrivals and c parallel servers of general service time. Exact methods for the steady state distributions of the $M/M/c$ queue and $M/D/c$ queue with constant service times are given, e.g. see Tijms [18]. Algorithmic methods are presented for computing the stationary distributions in $M/PH/c$ queue with the phase-type (PH) distribution of service time [14, 16, 17]. However, the system in general is considered to be mathematically intractable. Much effort has been spent on approximation for the system characteristics such as mean queue length and mean waiting time of the $M/G/c$ queue, cf. [3, 15, 10]. There are various approximations for the stationary distribution of the queue size. Halachmi and Franta [6], Kimura [9] and Choi and Shin [5] use the diffusion process for an approximation. Hokstad [7] derive the approximation formula by using the supplementary variable method. Tijms et al. [19] provide approximation formulae which are represented in terms of integration and recursive scheme. Miyazawa [13] presents the equations, called the basic equations from which approximation formula is derived in terms of generating functions. Choi et al [4] presents an approximation for the steady-state queue length distribution in $G/G/c/c+K$ using the first two moments of service time.

In the almost all literature mentioned above, the approximation results are compared with those of the $M/PH/c$ queue for the quality of approximation. The PH distributions

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used for comparisons are selected by the mean and the coefficient of variation (the ratio of the standard deviation to the mean) of the service time. There may be many PH distributions with the same mean and variance. So, unless the system performance measures are depend only on the first two moments of the service time, the approximations works well for some cases but not for other cases even though the service time distributions have the common mean and variance.

Objective of this paper has two folds. One is to investigate numerically the sensitivity of the system characteristics such as the mean and the standard deviation of the queue length and the blocking probability with respect to the moments of the service time. The other is to propose an approximation of the steady state distribution of the number of customers in $M/G/c$ queue. We show numerically that the mean and the standard deviation of the number of customers in queue are strongly affected by the third moment of the service time for some cases. Based on the sensitivity analysis, we approximate the service time distribution with PH distribution by matching the first three moments of the service times and use $M/PH/c$ queue for an approximation of $M/G/c$ queue.

In Section 2, the methods of the moment matching with PH distribution are briefly reviewed. In Section 3, we investigate numerically the sensitivity of some performance measures. Approximations with numerical results are presented in Section 4. Concluding remarks are given in Section 5.

2 The moment matching method to PH distributions

A distribution function $F(x)$ on $(0, \infty)$ is said to be of phase type with representation $(\boldsymbol{\alpha}, T)$ and denote it by PH$(\boldsymbol{\alpha}, T)$ if

$$F(x) = 1 - \boldsymbol{\alpha} \exp(Tx)\boldsymbol{e},$$

where $\boldsymbol{e}$ is the column m-vector whose components are all 1, $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \cdots, \alpha_m)$ is a probability distribution and $T = (t_{ij})$ is the $m \times m$ matrix with $t_{ii} < 0$, $1 \le i \le m$ and $t_{ij} \ge 0$, $i \ne j$, and $T\boldsymbol{e} \le \mathbf{0} (\ne \mathbf{0})$. For more details about PH-distribution, see [14, Chapter 2].

The phase type (PH) distribution is dense (in the sense of weak convergence) in the set of all probability distributions on $(0, \infty)$ (e.g. see Asmussen [1, page 84]). There are many moment matching methods for fitting the general distribution by the PH distributions cf. Bobbio et al. [2], Johnson and Taaffe [8] and Whitt [20]. In this section we review some moment matching methods to PH distributions.

Hyperexponential distribution : The hyperexponential distribution of order 2, denoted by $H_2(p; \mu_1, \mu_2)$ or simply H_2, has the probability density function of the form

$$f(t) = p\mu_1 e^{-\mu_1 t} + (1-p)\mu_2 e^{-\mu_2 t}.$$

If a positive random variable X with the first three moments m_1, m_2 and m_3 satisfies the squared coefficient of variation $C_s^2 = \frac{m_2 - m_1^2}{m_1^2} > 1$ and

$$m_1 m_3 > \frac{3}{2} m_2^2, \tag{1}$$

then the $H_2(p;\mu_1,\mu_2)$ density function can be fitted uniquely by the parameters (see [20] or [18])

$$\mu_{1,2} = \frac{1}{2}\left(a_1 \pm \sqrt{a_1^2 - 4a_2}\right), \quad p = \frac{\mu_1(1-\mu_2 m_1)}{\mu_1 - \mu_2}, \tag{2}$$

where

$$a_2 = \frac{6m_1^2 - 3m_2}{\frac{3}{2}m_2^2 - m_1 m_3}, \quad a_1 = \frac{1}{m_1}(1 + \frac{1}{2}m_2 a_2).$$

The requirement (1) holds for the gamma distribution, lognormal distribution and Weibul distribution with $C_s^2 > 1$.

Coxian distribution with Erlang node : Let $E_k(\mu)$ denote the Erlang distribution of order k with parameter μ. Denote by $CE_{k,j}(p;\mu_1,\mu_2)$ the composition of the mixture of $E_k(\mu_1)$ and $E_j(\mu_2)$ whose Laplace transform $f^*(s)$ is given by

$$f^*(s) = p\left(\frac{\mu_1}{\mu_1+s}\right)^k \left(\frac{\mu_2}{\mu_2+s}\right)^j + (1-p)\left(\frac{\mu_2}{\mu_2+s}\right)^j, \ s \geq 0.$$

Bobbio et al. [2] present explicit method to fit the first three moments of a positive random variable by $CE_{1,j}(p;\mu_1,\mu_2)$ and $CE_{k,1}(p;\mu_1,\mu_2)$. The formulae for determining the parameters in [2] are so complicated and are omitted here.

Mixture of Erlang distributions of common order : Johnson and Taaffe [8] provide a method that a mixture $E_{k,k}(p;\mu_1,\mu_2)$ of two Erlang distributions $E_k(\mu_1)$ and $E_k(\mu_2)$ with probability density function

$$f(t) = p\mu_1 \frac{(\mu_1 t)^{k-1}}{(k-1)!} e^{-\mu_1 t} + (1-p)\mu_2 \frac{(\mu_2 t)^{k-1}}{(k-1)!} e^{-\mu_2 t}$$

can fit the first three moments m_1, m_2 and m_3 of a positive random variable X. The parameters are given by

$$\mu_{1,2}^{-1} = \frac{1}{2a}\left(-b \pm \sqrt{b^2 - 4ac}\right), \quad p = \frac{\mu_1 - \mu_1\mu_2 m_1/k}{\mu_2 - \mu_1},$$

where

$$\begin{aligned} a &= k(k+2)m_1 y, \ b = -\left(kx + \frac{k(k+2)}{k+1}y^2 + (k+2)m_1^2 y\right), \ c = m_1 x, \\ y &= m_2 - \left(\frac{k+1}{k}\right)m_1^2, \ x = m_1 m_3 - \left(\frac{k+2}{k+1}\right)m_2^2. \end{aligned}$$

3 Sensitivity of $M/G/c$ queue

Consider the $M/G/c$ queue where customers arrive according to a Poisson process with rate λ and the service times of the customers are independent and identically distributed. Let S be the service time of a customer whose distribution function is $G(x)$ and assume that $G(0) = 0$, $m_k = E(S^k) < \infty$, $k = 1,2,3$. We also assume $\rho = \frac{\lambda m_1}{c} < 1$ for the stability of the system. Let X_s and X_q be the number of busy servers and customers waiting

Table 1: Sensitivity of P_0 and $P_B$in $M/G/3$ queue with $m_1 = 1.0$

				ρ			
G	C_s^2	m_3	m_4	0.3	0.5	0.7	0.9
				$P_0(M)$ and $P_0(M)-P_0(G)$			
M	1.0	6.0	24.0	0.4035	0.2105	0.0957	0.0249
E_2	0.5	3.0	7.5	0.0006	0.0015	0.0018	0.0009
$CE_{2,1}$	0.5	3.0	13.1	0.0006	0.0016	0.0020	0.0010
$CE_{1,3}$	0.5	10.0	454.1	0.0009	0.0021	0.0026	0.0013
H_2	2.0	18.0	162.0	-0.0008	-0.0024	-0.0029	-0.0015
H_2	2.0	18.0	153.1	-0.0010	-0.0031	-0.0040	-0.0021
H_2	2.0	100.0	11095.3	-0.0001	-0.0007	-0.0009	-0.0004
$CE_{1,2}$	5.0	300.0	24533.6	-0.0006	-0.0026	-0.0039	-0.0016
H_2	5.0	300.0	26724.0	-0.0008	-0.0031	-0.0042	-0.0019
H_2	5.0	60.0	804.0	-0.0007	-0.0022	-0.0029	-0.0017
				$P_B(M)$ and $P_B(M)-P_B(G)$			
M	1.0	6.0	24.0	0.0700	0.2368	0.4923	0.8171
E_2	0.5	3.0	7.5	0.0009	0.0030	0.0043	0.0026
$CE_{2,1}$	0.5	3.0	13.1	0.0009	0.0032	0.0046	0.0028
$CE_{1,3}$	0.5	10.0	454.1	0.0013	0.0043	0.0061	0.0037
H_2	2.0	18.0	162.0	-0.0012	-0.0042	-0.0061	-0.0034
H_2	2.0	18.0	153.1	-0.0016	-0.0055	-0.0076	-0.0044
H_2	2.0	100.0	11095.3	-0.0002	-0.0010	-0.0017	-0.0005
$CE_{1,2}$	5.0	300.0	24533.6	-0.0006	-0.0033	-0.0061	-0.0029
H_2	5.0	300.0	26724.0	-0.0011	-0.0046	-0.0076	-0.0038
H_2	5.0	60.0	804.0	-0.0018	-0.0061	-0.0091	-0.0044

in queue of the $M/G/c$ queue in steady state, respectively and set $L_s = \mathbb{E}(X_s), L_q = \mathbb{E}(X_q)$. By σ_s and σ_q, denote the standard deviation of X_s and X_q, respectively. It follows from Little's formula that $L_s = \lambda m_1$ does not depend on the second or the higher moments of the service time. For the single server case, it can be easily seen from Pollaczek-Khinchin transform equation, c.f. [12], that L_q depend only on the arrival rate λ and the first two moments m_1 and m_2 of the service time and σ_q is determined by λ and m_k, $k = 1,2,3$. It is also known that the steady state distribution of the number of busy servers in $M/G/\infty$ queue is given in terms of λ and m_1 (e.g. [18]).

In this section, we investigate numerically how σ_s, L_q, σ_q, $P_0 = \mathbb{P}(X_s = 0)$ and the blocking probability $P_B = \mathbb{P}(X_s = c)$ are affected by the moments of the service time. For this one, we fix $m_1 = 1.0$ and consider the three cases of squared coefficient of variation of the service time $C_s^2 = 0.5, 2.0, 5.0$.

In Tables 1-2, the values of P_0, P_B and σ_s are compared with those of the system with exponential service time (M). In Table 1, $P_0(G)$ and $P_B(G)$ denote the P_0 and P_B, respectively, of the system with service time distribution G and $\sigma_s(G)$ in Table 2 is defined analogously. The numerical results of the row corresponding to M in Tables 1-2 present the values of $P_0(M)$, $P_B(M)$ and $\sigma_s(M)$, respectively and other results are the difference between the results of service times M and G. Tables 1-2 shows that P_0, P_B and σ_s are affected weakly by the second or the higher moments of the service time which is expected from the results for the system with $c = 1$ and $c = \infty$. The effects of the moments of the

Table 2: Sensitivity of σ_s in $M/G/3$ queue with $m_1 = 1.0$ ($\sigma_s(M)$ and $\sigma_s(M) - \sigma_s(G)$)

				ρ			
G	C_s^2	m_3	m_4	0.3	0.5	0.7	0.9
M	1.0	6.0	24.0	0.9149	1.0699	1.0325	0.7028
E_2	0.5	3.0	7.5	0.0016	0.0042	0.0060	0.0049
$CE_{2,1}$	0.5	3.0	13.1	0.0018	0.0045	0.0064	0.0053
$CE_{1,3}$	0.5	10.0	454.1	0.0024	0.0061	0.0085	0.0071
H_2	2.0	18.0	162.0	-0.0022	-0.0061	-0.0087	-0.0070
H_2	2.0	18.0	153.1	-0.0039	-0.0080	-0.0114	-0.0092
H_2	2.0	100.0	11095.3	-0.0004	-0.0016	-0.0025	-0.0017
$CE_{1,2}$	5.0	300.0	24533.6	-0.0013	-0.0055	-0.0096	-0.0064
H_2	5.0	300.0	26724.0	-0.0020	-0.0071	-0.0113	-0.0081
H_2	5.0	60.0	804.0	-0.0027	-0.0077	-0.0115	-0.0101

Table 3: Effects of service time to L_q in $M/G/3$ queue with $m_1 = 1.0$

				ρ			
G	C_s^2	m_3	m_4	0.3	0.5	0.7	0.9
E_2	0.5	3.0	7.5	0.0240	0.1844	0.8778	5.5440
$CE_{2,1}$	0.5	3.0	13.1	0.0241	0.1846	0.8784	5.5449
$CE_{1,3}$	0.5	10.0	454.1	0.0222	0.1711	0.8379	5.4830
H_2	2.0	18.0	162.0	0.0385	0.3228	1.6420	10.884
H_2	2.0	18.0	153.1	0.0371	0.3165	1.6263	10.855
H_2	2.0	100.0	11095.3	0.0311	0.2588	1.4331	10.589
$CE_{1,2}$	5.0	300.0	24533.6	0.0329	0.3456	2.3582	20.242
H_2	5.0	300.0	26724.0	0.0382	0.3791	2.4719	20.445
H_2	5.0	60.0	804.0	0.0814	0.6692	3.3428	21.869

service time distribution to L_q and σ_q are investigated in Tables 3-4. Tables 3 shows that L_q is sensitive to the second and the third moments for small values of ρ and the sensitivity to the third moment decreases as ρ increases. We can see from Table 4 that σ_q depend strongly on the third moment m_3.

It follows from Tables 1-4 that the approximation of the distribution of X_s in $M/G/c$ queue using $M/M/c$ queue seems to be adequate for practical purpose but the third moment m_3 should be considered for an accurate approximation of L_q and σ_q.

4 Approximations

In this section we describe the approximation procedure and make some numerical comparisons for $M/G/3$ queue. Two service time distributions, Weibul distribution Weib(α, β) and lognormal distribution LN(μ, σ^2) are considered. For an approximation, we first choose an appropriate PH distribution by fitting the first three moments of the service time and then compute the performance characteristics of the approximating system. In order to fit the first three moments of the service time distribution with $C_s^2 < 1$, we adopt the method in Bobbio et al. [2] and for $C_s^2 > 1$, the formula (2) is used.

Approximations are compared with the simulation results in Tables 5-6. The performance measures for the approximating systems are computed by the matrix geometric

Table 4: Effects of service time to σ_q in $M/G/3$ queue with $m_1 = 1.0$

				ρ			
G	C_s^2	m_3	m_4	0.3	0.5	0.7	0.9
E_2	0.5	3.0	7.5	0.1967	0.6487	1.7642	6.9935
$CE_{2,1}$	0.5	3.0	13.1	0.1967	0.6488	1.7642	6.9935
$CE_{1,3}$	0.5	10.0	454.1	0.1857	0.6163	1.8152	7.7224
H_2	2.0	18.0	162.0	0.2953	1.1122	3.3341	13.956
H_2	2.0	18.0	153.1	0.2929	1.1110	3.3353	13.957
H_2	2.0	100.0	11095.3	0.2449	0.9557	3.6256	17.847
$CE_{1,2}$	5.0	300.0	24533.6	0.3285	1.7477	7.0468	33.264
H_2	5.0	300.0	26724.0	0.3423	1.7516	7.0343	33.273
H_2	5.0	60.0	804.0	0.5445	2.1025	6.3533	27.151

method [14]. Simulation models are developed with ARENA. Simulation run time is set to 80,000 unit times including 20,000 unit times of warm-up period, where the expected value of service time is one unit time. We used different random number streams for the distributions of inter-arrival times, service times. Ten replications are conducted for each case and the average value and the half length of 95% confidence interval are obtained.

The probability density function of Weibul distribution Weib(α,β) is given by

$$f(x) = \frac{\beta}{\alpha}\left(\frac{x}{\beta}\right)^{\alpha-1} \exp\left[-\left(\frac{x}{\beta}\right)^{\alpha}\right],\ x > 0$$

and the kth moment m_k and the squared coefficient of variations C_s^2 are given by

$$m_k = \frac{k\beta^k}{\alpha}\Gamma\left(\frac{1}{\alpha}\right),\ k = 1,2,\cdots,\quad C_s^2 = \frac{2\alpha}{\Gamma\left(\frac{1}{\alpha}\right)} - 1.$$

Thus Weib(α,β) is uniquely determined by the first two moments m_1 and m_2 or equivalently m_1 and C_s^2. If $m_1 = 1$, then $\beta = \alpha/\Gamma(1/\alpha)$ and

$$m_k = \frac{k}{2}(1+C_s^2)^{k-1},\ k = 1,2,\cdots.$$

In Table 5, for an approximation of Weib(α,β) with the (m_1, C_s^2) pair $(1.0, 0.5)$, $(1.0, 2.0)$ and $(1.0, 5.0)$ we respectively use the following distributions

$CE_{2,1}(0.751282; 2.88098, 2.09007)$ with $\hat{m}_4 = 11.1086$ $(m_4 = 6.79478)$,
$H_2(0.658726; 2.0365, 0.504441)$ with $\hat{m}_4 = 127.414$ $(m_4 = 136.423)$,
$H_2(0.908248; 1.8165, 0.183503)$ with $\hat{m}_4 = 1944.0$ $(m_4 = 2520.0)$,

where $\hat{m}_4$ and m_4 are the fourth moments of approximating distribution and Weib(α,β) distribution, respectively.

The probability density function of lognormal distribution LN(μ,σ^2) is given by

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma x}\exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right),\ x > 0$$

and the kth moment m_k and the squared coefficient of variations C_s^2 are given by

$$m_k = \exp\left(k\mu + \frac{1}{2}k^2\sigma^2\right),\ k = 1,2,\cdots,\quad C_s^2 = \exp(\sigma^2) - 1.$$

Thus $\mathrm{LN}(\mu,\sigma^2)$ is uniquely determined by m_1 and C_s^2. If $m_1 = 1$, then $\mu = -\sigma^2/2$ and

$$m_k = \frac{k}{2}(1 + C_s^2)^{k(k-1)/2},\ k = 1,2,\cdots.$$

In Table 6, for an approximation of $\mathrm{LN}(\mu,\sigma^2)$ with the (m_1, C_s^2) pair $(1.0, 0.5)$, $(1.0, 2.0)$ and $(1.0, 5.0)$ we respectively use the following distributions

$$\begin{array}{lll} \mathrm{CE}_{1,3}(0.116747; 0.950128, 3.42026) & \text{with} & \hat{m}_4 = 11.2631 \quad (m_4 = 11.3906), \\ \mathrm{H}_2(0.971405; 1.13807, 0.195262) & \text{with} & \hat{m}_4 = 485.994 \quad (m_4 = 729.0), \\ \mathrm{H}_2(0.99075; 1.15827, 0.06395) & \text{with} & \hat{m}_4 = 13284.0 \quad (m_4 = 46656.0), \end{array}$$

where $\hat{m}_4$ and m_4 are the fourth moments of approximating PH distribution and lognormal distribution, respectively.

For the accuracy of the simulation, the comparisons with the exact results for L_s are presented in Tables 5-6. Numerical results show that the approximations work well for small value of C_s^2 and for large C_s^2, the approximations improves as ρ increases.

5 Conclusions

We have investigated numerically the effects of the moments of the service time to the performance measures related with the number X_s of busy servers and the number X_q of customers in queue in $M/G/c$ queue. Numerical experiments show that the effect of the third moment of the service time to the mean queue length L_q is not negligible for ρ small and C_s^2 large and the standard deviation σ_q of X_q is strongly affected by not only the second moment but also the third moment of the service time, while the distribution of X_s is less sensitive to the second or higher moment of the service time than X_q.

We approximate the $M/G/c$ queue by $M/PH/c$ queue where the PH distribution is affected by the first three moments of the service time. Numerical experiments leads to approximations that are significantly accurate for wide range of service times.

The method to approximate the multi server queue by fitting the service time with PH distributions requires relatively long computation times, which often restricts the number of servers and the number of phases of PH distribution. However, the many distributions with C_s^2 not close to 0 arising in practical situation can be fitted by the PH distribution with the moderate number of phases which reduces the computational problem. For example, the many distributions with $C_s^2 > 1$ can be fitted by the H_2 distribution and the matrix components of the generator of the quasi-birth-and-death process corresponding to the system $M/H_2/c$ queue is $c+1$. This research is expected to be a preliminary step to apply the method to the variants of the basic $M/G/c$ queue like the queue with retrials.

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Table 5: Approximation of $M/Weib(\alpha,\beta)/3$ queue with $m_1 = 1.0$

C_s^2	ρ		L_s	σ_s	L_q	σ_q	P_0	P_B
0.5	0.3	App	0.9000	0.9130	0.0243	0.1980	0.4028	0.0690
		Sim	0.8990	0.9121	0.0240	0.1968	0.4029	0.0687
		(c.i.)	±0.0033	±0.0016	±0.0005	±0.0035	±0.0015	±0.0007
	0.5	App	1.5000	1.0650	0.1858	0.6502	0.2087	0.2334
		Sim	1.5001	1.0653	0.1860	0.6547	0.2087	0.2338
		(c.i.)	±0.0045	±0.0015	±0.0047	±0.0192	±0.0017	±0.0015
	0.7	App	2.1000	1.0256	0.8811	1.7625	0.0935	0.4874
		Sim	2.1005	1.0266	0.8886	1.7737	0.0938	0.4886
		(c.i.)	±0.0068	±0.0025	±0.0178	±0.0454	±0.0012	±0.0030
	0.9	App	2.7000	0.6971	5.5497	6.9821	0.0238	0.8142
		Sim	2.6989	0.6987	5.5847	7.0502	0.0241	0.8138
		(c.i.)	±0.0080	±0.0076	±0.2156	±0.3667	±0.0007	±0.0050
2.0	0.3	App	0.9000	0.9170	0.0399	0.3014	0.4042	0.0712
		Sim	0.9023	0.9182	0.0416	0.3060	0.4035	0.0717
		(c.i.)	±0.0055	±0.0033	±0.0021	±0.0123	±0.0022	±0.0014
	0.5	App	1.5000	1.0758	0.3303	1.1206	0.2127	0.2410
		Sim	1.5000	1.0746	0.3358	1.1201	0.2123	0.2402
		(c.i.)	±0.0083	±0.0015	±0.0053	±0.0216	± 0.0023	±0.0028
	0.7	App	2.1000	1.0410	1.6613	3.3200	0.0984	0.4984
		Sim	2.1008	1.0385	1.6538	3.2444	0.0975	0.4974
		(c.i.)	±0.0080	±0.0026	±0.0485	±0.1331	±0.0014	±0.0037
	0.9	App	2.7000	0.7098	10.919	13.862	0.0263	0.8205
		Sim	2.6999	0.7077	10.897	13.759	0.0259	0.8196
		(c.i.)	±0.0072	±0.0079	±0.7291	±1.3317	±0.0009	±0.0040
5.0	0.3	App	0.9000	0.9197	0.0612	0.4841	0.4051	0.0728
		Sim	0.9038	0.9209	0.0707	0.4964	0.4038	0.0730
		(c.i.)	±0.0082	±0.0042	±0.0042	±0.0281	±0.0032	±0.0019
	0.5	App	1.5000	1.0838	0.5616	2.0338	0.2157	0.2466
		Sim	1.5002	1.0805	0.6034	2.0042	0.2144	0.2445
		(c.i.)	±0.0116	±0.0014	±0.0232	±0.1024	±0.0034	±0.0036
	0.7	App	2.1000	1.0529	3.0630	6.5095	0.1026	0.5067
		Sim	2.1002	1.0465	3.1076	6.2164	0.1001	0.5027
		(c.i.)	±0.0100	±0.0031	±0.1615	±0.4208	±0.0017	±0.0047
	0.9	App	2.7000	0.7196	21.360	28.058	0.0286	0.8253
		Sim	2.6995	0.7145	21.521	27.961	0.0273	0.8226
		(c.i.)	±0.0108	±0.0110	±2.4488	±4.3256	±0.0011	±0.0060

Table 6: Approximation of $M/LN(\alpha,\beta)/3$ queue with $m_1 = 1.0$

C_s^2	ρ		L_s	σ_s	L_q	σ_q	P_0	P_B
0.5	0.3	App	0.9000	0.9133	0.0233	0.1933	0.4030	0.0691
		Sim	0.8988	0.9124	0.0230	0.1916	0.4032	0.0688
		(c.i.)	±0.0033	±0.0016	±0.0005	±0.0035	±0.0015	±0.0007
	0.5	App	1.5000	1.0658	0.1810	0.6440	0.2092	0.2338
		Sim	1.4999	1.0660	0.1811	0.6485	0.2092	0.2340
		(c.i.)	±0.0047	±0.0015	±0.0050	±0.0226	±0.0017	±0.0017
	0.7	App	2.1000	1.0266	0.8694	1.7703	0.0940	0.4880
		Sim	2.1005	1.0275	0.8770	1.7847	0.0943	0.4890
		(c.i.)	±0.0072	±0.0026	±0.0198	±0.0544	±0.0013	±0.0032
	0.9	App	2.7000	0.6978	5.5294	7.0386	0.0240	0.8144
		Sim	2.6990	0.6992	5.5730	7.1301	0.0244	0.8141
		(c.i.)	±0.0082	±0.0078	±0.2390	±0.4443	±0.0006	±0.0051
2.0	0.3	App	0.9000	0.9165	0.0349	0.2747	0.4041	0.0709
		Sim	0.9026	0.9188	0.0363	0.2812	0.4036	0.0720
		(c.i.)	±0.0059	±0.0033	±0.0019	±0.0127	±0.0024	±0.0014
	0.5	App	1.5000	1.0748	0.2990	1.0737	0.2126	0.2401
		Sim	1.5004	1.0760	0.3028	1.0652	0.2130	0.2411
		(c.i.)	±0.0079	±0.0013	±0.0068	±0.0339	±0.0020	±0.0027
	0.7	App	2.1000	1.0396	1.5756	3.3942	0.0982	0.4972
		Sim	2.1004	1.0412	1.5598	3.2883	0.0987	0.4987
		(c.i.)	±0.0078	±0.0025	±0.0603	±0.2128	±0.0014	±0.0037
	0.9	App	2.7000	0.7082	10.772	14.461	0.0261	0.8197
		Sim	2.6996	0.7097	10.805	14.666	0.0264	0.8201
		(c.i.)	±0.0076	±0.0081	±0.9843	±2.3320	±0.0009	±0.0043
5.0	0.3	App*	0.9000	0.9176	0.0418	0.3735	0.4046	0.0715
		Sim	0.9043	0.9216	0.0554	0.4190	0.4038	0.0734
		(c.i.)	±0.0085	±0.0044	±0.0035	±0.0296	±0.0034	±0.0020
	0.5	App	1.5000	1.0789	0.4174	1.8319	0.2143	0.2428
		Sim	1.5008	1.0828	0.4972	1.7998	0.2153	0.2461
		(c.i.)	±0.0106	±0.0013	± 0.0195	± 0.1147	± 0.0029	± 0.0034
	0.7	App	2.1000	1.0465	2.6156	6.8781	0.1009	0.5018
		Sim	2.0997	1.0499	2.8525	6.8082	0.1016	0.5045
		(c.i.)	±0.0101	±0.0029	±0.3057	±1.4804	±0.0017	±0.0049
	0.9	App	2.7000	0.7134	20.624	31.331	0.0273	0.8221
		Sim	2.6989	0.7176	21.579	32.641	0.0281	0.8234
		(c.i.)	±0.0117	±0.0119	±3.7717	±9.3426	±0.0012	±0.0067