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Certainty Equivalent in Portfolio Management

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Abstract In portfolio selection, strategies on an efficient frontier have been regarded as non-dominated solutions because of the compensation to each increase unit of risk, and a rational decision maker has to consider other supplementary decision rules. This paper proposes an approach that helps a rational decision maker identify the best candidate strategy on an efficient frontier by taking the concept of certainty equivalent from decision analysis. It is shown that by integrating the efficient frontier and an approximation of certainty equivalent based on the widely used exponential utility function in the same coordinate plane, we are able to derive an analytical solution to the optimal strategy, and thus develop an efficient selecting procedure that can significantly reduce the computational load as a result of the necessary comparisons between only one or two candidate portfolios.

Keywords portfolio selection; efficient frontier; certainty equivalent; quadratic programming

1 Introduction

Modern portfolio management has been heavily motivated by Markowitz's portfolio theory; see [e.g. [8, 9]], in which the risk and return are measured by variance and expected rate of return, respectively. In portfolio selection, one of the most important assumptions is that a decision maker must be rational, namely, he is supposed to choose the portfolio that

- minimizes the risk given a fixed level of expected return; or
- maximizes the expected return given a fixed level of risk.

According to the portfolio theory, given a set of risky assets, a rational decision maker is able to sketch out an efficient frontier. Each portfolio lying on the efficient frontier can be regarded as an efficient portfolio in the sense that none of these strategies is dominated by others because of the compensation to each increase unit of risk. As a result, the portfolio theory itself seems far from enough for a decision maker to reach an exclusive decision. In order to discriminate the efficient portfolios and construct a preference order, a decision maker has to consider other supplementary decision rules. Since the principle of maximizing expected utility (PMEU), as the most widely adopted decision rule in decision theory;see [e.g. [1, 3, 5, 6]], has been highly criticized for many different aspects, alternative as well as supplementary decision rules have been proposed, among which the concept of certainty equivalent (CE) in decision analysis plays an important role; see [e.g. [11, 12]].

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This paper proposes an approach that helps a rational decision maker identify the best candidate strategy on an efficient frontier by taking the concept of CE from decision analysis. It is shown that by integrating the efficient frontier and an approximation of CE based on the widely used exponential utility function in the same mean-variance coordinate plane, we can derive an analytical solution to the optimal strategy, with which an efficient selecting procedure can be developed. This approach can be regarded as an improvement to the selecting strategy proposed in [12]. The corresponding computational load can be significantly reduced as a result of the fixed number of comparisons between only one or two candidate portfolios. An important assumption of the proposed approach is that we exclude the existence of risk-less asset in portfolio management because otherwise, the analytical solution we derive herein would appear in another form; see [e.g. [2]].

The following section roughly reviews the background knowledge about the efficient frontier in portfolio management and the concept of CE in decision analysis. The third section presents an efficient selecting procedure being able to indicate the best candidate strategy on an efficient frontier, which is followed by a numerical example for illustration. The final section concludes this paper.

2 Efficient Frontier and Certainty Equivalent

2.1 Efficient Frontier

Suppose there are at least two different risky assets in a portfolio. We use an n-dimensional ($n \geq 2$) column vector $r = (r_1, \ldots, r_n)'$ to represent their expected rates of return, and an n-dimensional column vector $w = (w_1, \ldots, w_n)'$ to represent the corresponding weights of assets, for which $\sum_{i=1}^{n} w_i = 1$ must hold.

The risk of a single asset X can be measured by its variance denoted by $\sigma_X^2 = E(X - E[X])^2$. Apparently, the greater its variance, the more risky the asset is. Besides, when it comes to the risk of a portfolio, we need to consider the covariance denoted by $Cov(X,Y) = \sigma_{XY} = E[(X - E[X])(Y - E[Y])]$ between two different risky assets X and Y. Extending the concept of covariance into n risky assets, we can build up a symmetrical covariance matrix V, since $\sigma_{XY} = \sigma_{YX}$, to measure the risk of a portfolio, and have the following quadratic programming (QP) model

$$\begin{array}{ll} \min & \sigma^2 = w'Vw \\ \text{s.t.} & w'r = r_p; w'I = 1 \end{array} \tag{1}$$

In (1), we try to minimize the risk of a portfolio, $\sigma^2 = w'Vw$, given the level of expected rate of return $w'r = r_p$ and the normalization constraint $w'I = 1$ where I is an n-dimension column vector $(1, 1, \ldots, 1)'$. It should be noted that the covariance matrix V in (1) is supposed to be positive definite.

To solve (1), we can obtain the following function $L(w, \lambda_1, \lambda_2)$ by using the Lagrange method

$$L(w, \lambda_1, \lambda_2) = -w'Vw + \lambda_1(w'r - r_p) + \lambda_2(w'I - 1) \tag{2}$$

By setting the first-order derivatives of (2) to zeros with respect to w, λ_1 and λ_2, respectively, we can obtain

$$\begin{aligned} &\partial L(w,\lambda_1,\lambda_2)/\partial w = 2Vw - \lambda_1 r - \lambda_2 \mathit{1} = \mathbf{0} \\ &\partial L(w,\lambda_1,\lambda_2)/\partial \lambda_1 = r_p - w'r = 0 \\ &\partial L(w,\lambda_1,\lambda_2)/\partial \lambda_2 = 1 - w'\mathit{1} = 0 \end{aligned} \tag{3}$$

Thereafter, by pre-multiplying the first equation of (3) by V^{-1} due to the positive definiteness of the covariance matrix V, we can obtain

$$w = \frac{1}{2}V^{-1}(\lambda_1 r + \lambda_2 \mathit{1}) = \frac{1}{2}V^{-1}\begin{bmatrix} r & \mathit{1} \end{bmatrix}\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} \tag{4}$$

Besides, with the combination of the second and third equations of (3), we can readily have

$$\begin{bmatrix} r & \mathit{1} \end{bmatrix}' w = \begin{bmatrix} r_p \\ 1 \end{bmatrix} \tag{5}$$

Finally, by pre-multiplying equation (4) by $[r\ \mathit{1}]'$ and combining it with equation (5), we can have

$$\begin{bmatrix} r_p \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} r & \mathit{1} \end{bmatrix}' V^{-1}\begin{bmatrix} r & \mathit{1} \end{bmatrix}\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} \tag{6}$$

Now suppose $a = r'V^{-1}r, b = r'V^{-1}\mathit{1}, c = \mathit{1}'V^{-1}\mathit{1}$ and $d = ac - b^2$. We are able to rewrite d in its matrix form as

$$d = \begin{bmatrix} a & b \\ b & c \end{bmatrix} = \begin{bmatrix} r'V^{-1}r & r'V^{-1}\mathit{1} \\ r'V^{-1}\mathit{1} & \mathit{1}'V^{-1}\mathit{1} \end{bmatrix} = \begin{bmatrix} r & \mathit{1} \end{bmatrix}' V^{-1}\begin{bmatrix} r & \mathit{1} \end{bmatrix} \tag{7}$$

Therefore, equation (6) can be simplified as

$$\begin{bmatrix} r_p \\ 1 \end{bmatrix} = \frac{1}{2}d\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} \tag{8}$$

So

$$w = V^{-1}\begin{bmatrix} r & \mathit{1} \end{bmatrix} d^{-1}\begin{bmatrix} r_p \\ 1 \end{bmatrix} \tag{9}$$

Thus, we have

$$\sigma_p^2 = w'Vw = \begin{bmatrix} r_p & 1 \end{bmatrix} d^{-1}\begin{bmatrix} r & \mathit{1} \end{bmatrix}' V^{-1}VV^{-1}\begin{bmatrix} r & \mathit{1} \end{bmatrix} d^{-1}\begin{bmatrix} r_p \\ 1 \end{bmatrix} \tag{10}$$

It should be noted that

$$\begin{bmatrix} r & \mathit{1} \end{bmatrix}' V^{-1}VV^{-1}\begin{bmatrix} r & \mathit{1} \end{bmatrix} = d$$

Consequently, we have

$$\sigma_p^2 = \begin{bmatrix} r_p & 1 \end{bmatrix} d^{-1} \begin{bmatrix} r_p \\ 1 \end{bmatrix} = \begin{bmatrix} r_p & 1 \end{bmatrix} \frac{1}{ac-b^2} \begin{bmatrix} c & -b \\ -b & a \end{bmatrix} \begin{bmatrix} r_p \\ 1 \end{bmatrix} \tag{11}$$

In algebra form, it is

$$\sigma_p^2 = \frac{c}{d}(r_p^2 - \frac{2b}{c} r_p + \frac{a}{c}) \tag{12}$$

In fact, equation (12) depicts a parabola in a coordinate plane with σ^2 as the abscissa and r_p as the ordinate. The upper part of this parabola is defined as an efficient frontier. Portfolios lying on the efficient frontier are non-dominated due to the compensation in the expected rate of return to each increase unit of risk.

2.2 Certainty Equivalent

According to the preference theory, there are basically three types of risk attitudes for a decision maker: risk-averse, risk-neutral and risk-seeking, among which risk-averse is the most common one and assumed to be the proper risk attitude for a rational decision maker.

The utility functions representing the risk-averse attitude are usually assumed to be differentiable, increasing and concave, e.g., quadratic functions, exponential functions, etc. One of the widely adopted such utility functions is the exponential utility function, $U(x) = -e^{-x/R}$, where R is the risk tolerance; x is the wealth one possesses; and e is the base of natural logarithms. Larger R indicates less risk-averse of a rational decision maker.

The preference of a rational decision maker can be measured by CE defined as "the amount of money that is equivalent in your mind to a given situation that involves uncertainty" [1]. For a typical risk-averse decision maker, CE is usually less than the expected value. Moreover, a rational decision maker should prefer the strategy with a greater CE.

Based on the exponential utility function, an approximation of CE is

$$CE \approx \mu - \sigma^2/2R \tag{13}$$

where μ is the expected rate of return of an efficient portfolio; σ^2 is the corresponding variance; and R is the risk tolerance from a decision maker [7, 10]. It should be noted that in the mean-variance coordinate plane, this approximation describes CE as a straight line, and therefore facilitates the development of an effective selecting procedure to construct a preference order for the candidate portfolios on an efficient frontier.

3 Portfolio Selection

To discriminate the candidate portfolios, we integrate the parabola (12) and approximation (13) into the same mean-variance coordinate plane.

Given the risk tolerance R from a rational decision maker, we can observe in Figure 1 that the highest obtainable CE exists when the straight line with a fixed slope and the parabola has only one intersection point, namely, when the straight line goes through the

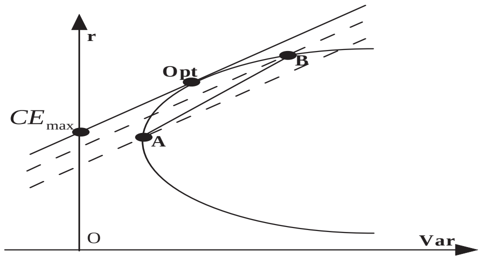

Figure 1: Decision-making with Certainty Equivalent

point “Opt”. Therefore, in order to derive an analytical solution to the optimal strategy on the efficient frontier, we need to figure out μ and σ^2 at this specific point.

Suppose $x = \sigma_p^2$, $y = r_p$

$$x = \frac{c}{d}(y^2 - \frac{2b}{c}y + \frac{a}{c}) \Longrightarrow y = \sqrt{\frac{d}{c}x - \frac{ac-b^2}{c^2}} + \frac{b}{c} \tag{14}$$

The first-order derivative of y is

$$y' = \frac{d}{2\sqrt{dcx - ac + b^2}} \tag{15}$$

Let $y' = 1/2R$, and we can get

$$x = R^2 a + \frac{1 - R^2 b^2}{c} \Longrightarrow \sigma_p^2 = R^2 a + \frac{1 - R^2 b^2}{c} \tag{16}$$

It can be observed in Figure 1 that except for the optimal strategy, the straight line (13) is a secant line rather than a tangent line to an efficient frontier. Consequently, the CE values of the candidates, interpreted as the intercepts of parallel straight lines with a fixed slope crossing these candidate portfolios on the efficient frontier, can be readily measured. Apparently, the CE value of a candidate portfolio decreases as the straight line moves to the bottom right. Moreover, the efficient frontier can be divided into two parts by the optimal strategy. For the candidates lying on the right of the optimal strategy, the CE value decreases as the expected rate of return r_p increases; while for the candidates lying on the left of the optimal strategy, the CE value decreases as r_p decreases. Generally speaking, a candidate portfolio lying closer to the optimal strategy has a higher CE value, and thereby can be selected as the best candidate strategy.

With (14) and (16), we can obtain r_p of the optimal strategy, denoted by r_o. By comparing the known r_ps of all candidate portfolios on the efficient frontier with r_o, we can identify the best candidate strategy by taking the following approach:

1. If the r_ps of all candidates are greater than r_o, which means that all candidates lie on the right of the optimal strategy on the efficient frontier, a rational decision maker should choose the candidate with the smallest r_p.

2. If the r_ps of all candidates are less than r_o, which means that all candidates lie on the left of the optimal strategy on the efficient frontier, a rational decision maker should choose the candidate with the largest r_p.
3. If for two neighboring candidates with $r_{p1} < r_{p2}$, we have $r_{p1} < r_o < r_{p2}$, then we can narrow down the scope of choices to only two candidates. After calculating their r_ps and σ_p^2s, we are able to derive the slope of the straight line through these two candidates, i.e., $k = (r_{p2} - r_{p1})/(\sigma_{p2}^2 - \sigma_{p1}^2)$.
   - If $k < 1/2R$, a rational decision maker should choose the candidate with the expected rate of return r_{p1}.
   - If $k > 1/2R$, a rational decision maker should choose the candidate with the expected rate of return r_{p2}.
4. If for certain candidate, we have $r_p = r_o$, a rational decision maker should choose the candidate with the expected rate of return r_p.

For instance, the second case in condition 3 can be demonstrated by Figure 1 in which we take r_{p1}, r_{p2} and r_o as the expected rates of return for A, B, and Opt, respectively.

4 Numerical Example

We choose the monthly stock prices of five retail companies over March 1^{st}, 2007 to August 31^{st}, 2007; see Table 1, in order to construct portfolios with respect to various expected rates of return. The data is taken from their officially released financial reports.

	Safeway	Walgreen	Lowe's	Target	Wal-Mart
March	36.49	45.71	31.27	58.99	46.50
April	36.15	43.73	30.39	59.10	47.46
May	34.34	45.03	32.64	62.28	47.36
June	33.96	43.45	30.52	63.45	47.87
July	31.80	44.09	27.93	60.42	45.72
August	31.66	45.07	30.97	65.93	43.63

Table 1: Monthly Stock Prices of Five Retail Companies

Table 2 summarizes the basic information about the covariances and expected rates of return of these five retail companies.

	Safeway	Walgreen	Lowe's	Target	Wal-Mart	r_p
Safeway	4.2457	0.2285	1.3828	-3.8582	2.1528	34.07
Walgreen	0.2285	0.7865	0.6883	0.0258	-0.6064	44.51
Lowe's	1.3828	0.6883	2.3849	0.9514	0.4946	30.62
Target	-3.8582	0.0258	0.9514	7.4068	-2.1487	61.70
Wal-Mart	2.1528	-0.6064	0.4946	-2.1487	2.4696	46.42

Table 2: Covariance Matrix and Expected Rates of Return

On the efficient frontier, suppose we have five candidate portfolios denoted by str1 through str5 in Table 3 that shows their corresponding compositions, expected rates of return, and variances.

	Safeway	Walgreen	Lowe's	Target	Wal-Mart	r_p	σ_p^2
str1	0.21	1.29	-2.30	0.81	0.97	90	4.91
str2	0.22	1.38	-2.56	0.89	1.06	95	6.34
str3	0.23	1.47	-2.81	0.97	1.15	100	7.96
str4	0.24	1.56	-3.07	1.04	1.23	105	9.77
str5	0.24	1.65	-3.33	1.12	1.32	110	11.76

Table 3: Compositions of Five Candidate Portfolios

Given different levels of risk tolerance, Table 4 shows r_os and the corresponding best choices according to the proposed approach. For example, with $R = 0.15$, the theoretical optimal strategy ($r_o = 98.64$) is located between two candidate portfolios, namely, str2 and str3. After solving two QP programs by conducting (1), we can derive the corresponding r_ps and σ_p^2s for str2 and str3, respectively. Then we simply calculate the slope of the straight line through these two candidate portfolios, $k = (100-95)/(7.96-6.34) = 3.09 < 1/2R \approx 3.33$, and conclude that a rational decision maker should choose str2 in consistence with the first case in condition 3 in the proposed approach. Furthermore, with $R = 0.10$ and $R = 0.20$, it is unnecessary to carry out any more computations after we figure out the r_os according to condition 1 and 2 in the proposed approach, respectively.

	New Method		Traditional Method (CE Values)					
	r_o	Choice	str1	str2	str3	str4	str5	Choice
$R = 0.10$	83.73	str1	65.46	63.29	60.19	56.16	51.20	str1
$R = 0.15$	98.64	str2	73.64	73.86	73.46	72.44	70.80	str2
$R = 0.20$	113.54	str5	77.73	79.14	80.09	80.58	80.60	str5

Table 4: New Method versus Traditional Method

Nevertheless, with the traditional method, given different levels of risk tolerance, we need to compute the corresponding CEs according to the approximation (13) for each candidate portfolio, carry out the comparisons, and choose the candidate with the highest CE [12]; see Table 4. It should be noted that during this process, we have to figure out r_p and σ_p^2 for each candidate portfolio by conducting (1). We can observe in the rightmost column of Table 4 that the best choices are str1, str2 and str5 with $R = 0.10$, $R = 0.15$ and $R = 0.20$, respectively, identical to those indicated by the new method.

The numerical example we have discussed above can be illustrated by Figure 2, in which Opt1, Opt2 and Opt3 represent the optimal strategy with $R = 0.10$, $R = 0.15$ and $R = 0.20$, respectively.

The advantage of the proposed method in terms of the computational load is obvious. With the traditional method, given a risk tolerance level, we need to figure out all variances for candidates with different expected rates of return by conducting (1). Accordingly, we have to conduct (1) N times if we have N candidate portfolios on an efficient frontier. Nevertheless, with the new method, we just need to obtain r_o in accordance with (14) and (16) first, and then compare r_o with various expected rates of return of the candidate portfolios in order to find out the location of the optimal strategy. Thereafter, we need to further solve at most two more QP programs under condition 3, or no more computations

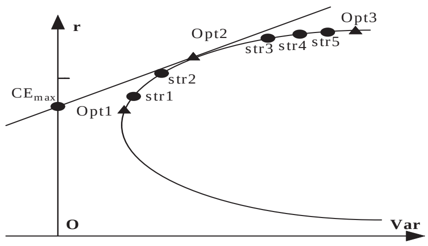

Figure 2: Illustration with Five Candidate Portfolios

under condition 1, 2 or 4, to identify the best candidate strategy. Consequently, we need to conduct (1) no more than three times and considerable computational efforts can be saved as the number of candidate portfolios increases.

5 Conclusion

This paper proposes a method for handling the decision-making process among the candidate portfolios lying on an efficient frontier by making use of the concept of CE from decision analysis. We have provided an analytical solution to the optimal strategy and developed an effective selecting procedure to help a decision maker identify the best candidate strategy.

The approach holds for the exponential utility function since the approximation of its CE only involves μ and σ^2, and can be integrated into the mean-variance coordinate plane. It is particularly suitable for the decision-making process with a large number of candidate portfolios on an efficient frontier due to the significant savings in the computational load.

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