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Some Properties of Semi-E-Convex Function and Semi-E-Convex Programming*

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Abstract In Ref 1,Yang shows that some of the results obtained in Ref. 2 on *E*-convex programming are incorrect, in Ref 3,Chen introduce semi-E-convex functions to correct the main results in Ref.2, in Ref 4,Duca and Lupsa show that the results obtained in Ref. 2 concerning the characterization of an *E*-convex function *f* in terms of its *E*-epigraph are incorrect. And give some characterizations of *E*-convex functions using two notion of epigraph ($epi_E(f)$ *and* $epi(f)$). In this note, some new properties of semi-*E*-convex functions are discussed, and new characterizations of semi-*E*-convex functions using a new epigraph of $epi^E(f)$ and slack 2-convex set are proposed, more new results of semi-E-convex programming are given.

Keywords *E*-convex sets; semi-*E*-convex functions; epigraphs; semi-E-convex programming; slack 2-convex sets

1 Introduction

The convexity of functions is important in the discussion of optimization and variation inequalities, to weak the convexity of functions attracted more attention of researchers, see [1-8]. Youness introduced the concepts of *E*- convex set and *E*-convex function in Ref. 2. Chen introduced the definitions of semi-E-convex function and quasi-semi- E-convex function in Ref. 3. For convenience, we recall these definitions. And give other related concepts, which is required in the later discussions.

Definition 1.1[2]. Let $E: R^n \to R^n$ be a function. A subset $M \subset R^n$ is said to be *E*-convex if

$$(1-t)E(x)+tE(y) \in M ,$$

for all $x, y \in M$ and all $t \in [0,1]$.

Definition 1.2[2]. Let *M* be a nonempty subset of R^n and let $E: R^n \to R^n$ be a function. A function $f: M \to R$ is said to be *E*-convex on *M* if *M* is *E*-convex and

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$$f((1-t)E(x)+tE(y)) \le (1-t)f(E(x))+tf(E(y)),$$

for all $x, y \in M$ and all $t \in [0,1]$.

Definition 1.3[3].. A function $f: M \to R$ is said to be semi- *E*-convex on a set $M \subset R^n$ iff there is a mapping $E: R^n \to R^n$ such that M is a *E*-convex set and

$$f((1-t)E(x)+tE(y)) \le (1-t)f(x)+tf(y)),$$

for each $x, y \in M$ and all $t \in [0,1]$.

Definition 1.4[3]. A function $f: M \to R$ is said to be quasi-semi- *E*-convex on a set $M \subset R^n$ iff there is a mapping $E: R^n \to R^n$ such that M is a *E*-convex set and

$$f((1-t)E(x)+tE(y)) \le \max\{f(x), f(y)\}$$

for each $x, y \in M$ and $t \in [0,1]$

Definition 1.5. A function $f: M \to R$ is said to be strictly quasi-semi-*E*-convex on a set $M \subset R^n$ iff there is a mapping $E: R^n \to R^n$ such that M is a *E*-convex set and for each $x, y \in M$ with $f(x) \ne f(y)$ we have

$$f((1-t)E(x)+tE(y)) < \max\{f(x), f(y)\}, \ \forall t \in (0,1).$$

Definition 1.6. A function $f: M \to R$ is said to be strongly quasi-semi-*E*-convex on a set $M \subset R^n$ iff there is a mapping $E: R^n \to R^n$ such that M is a *E*-convex set and for each $x, y \in M$ with $x \ne y$ we have

$$f((1-t)E(x)+tE(y)) < \max\{f(x), f(y)\}, \ \forall t \in (0,1).$$

Definition 1.7[3]. The function $f: M \to R$ is said to be pseudo-semi- *E*-convex on *E*-convex set $M \subset R^n$, if there exists a strictly positive function $b: R^n \times R^n \to R$ such that

$$f(x) < f(y) \Rightarrow f(tE(x)+(1-t)E(y)) \le f(y)+t(t-1)b(x,y)$$

for all $x, y \in M$, and $t \in [0,1]$.

In Ref. 2, the concepts of *E*-convex sets and *E*-convex functions are given, its properties are proposed, and the related results are used in the study of *E*-convex programming. Unfortunately, some of the results obtained in Ref. 2 are incorrect. Indeed, in Ref. 1 and Ref. 3, Yang and Chen shows that some of the results obtained in Ref. 2 on *E*-convex programming are incorrect respectively, but does not prove that the result which makes the connection between an *E*-convex function and its *E*-epigraph is incorrect. In Ref. 4, Duca and Lupsa show that the result obtained in Ref. 2 on the characterization of an *E*-convex function *f* in terms of its *E*-epigraph($E-e(f)$) is not true. And give some characterizations of *E*-convex function using notion of $epi_E(f)$ *and* epi (f). In this note, we discuss more new properties about semi-E-convex functions and give also some characterizations of semi-E-convex function using a new notion of epigraph (i.e. $epi^E(f)$), which is first introduced in the note. We propose some new results of semi-E-convex programming.

2 Some Properties of semi-*E*-Convex Functions

In this section, some relations between different notions about semi-*E*-convex functions and properties of semi-*E*-convex functions are given. And the characterizations of semi-*E*-convex functions in terms of a new notion of epigraph, $epi^E(f)$ is obtained.

If M is a nonempty subset of R^n and $E: M \to M$ and $f: M \to R$ are two functions, we consider the following three sets:

$$E - e(f) = \{(x,a) \in M \times R \mid f(E(x)) \le a\},$$

$$\text{epi}(f) = \{(x,a) \in M \times R \mid f(x) \le a\},$$

$$epi_E(f) = \{(z,a) \in E(M) \times R \mid f(z) \le a\},$$

$$epi^E(f) = \{(Ex,a) \in E(M) \times R \mid f(x) \le a\},$$

The four sets $E - e(f)$,epi(*f*), $epi_E(f)$ and $epi^E(f)$ are not equal. Obviously, we have $epi_E(f) \subset epi(f)$.

The following theorem proposes relations among different notions about semi-*E*-convex functions **Theorem 2.1.** A strongly quasi-semi- *E*-convex function on a set $M \subset R^n$ is also a strictly quasi-semi- *E*-convex function on the set M .

Theorem 2.2. Let M be a nonempty subset of R^n and let $f: M \to R$ and $E: R^n \to R^n$ be two functions. If *M* is an *E*-convex set and *f* is a semi-*E*-convex function on *M*, then $epi(f) \subset E - e(f)$.

Theorem 2.3. Let M be a nonempty subset of R^n and let $f: M \to R$ and $E: R^n \to R^n$ be two functions. If $epi(f) \subset E - e(f)$ and *f* is *E*-convex function on *M*. Then *f* is a semi-*E*-convex function on *M*.

Proof. For $x, y \in M$ and $t \in [0,1]$, $(x, f(x)), (y, f(y)) \in epi(f)$,thus

$$(x, f(x)), (y, f(y)) \in E - e(f),$$

which implies that $f(Ex) \le f(x)$, $f(Ey) \le f(y)$ as *f* is *E*-convex function on *M*, we have

$$\begin{aligned} f((1-t)E(x) + tE(y)) &\le (1-t)f(E(x)) + tf(E(y)) \\ &\le (1-t)f(x) + tf(y). \end{aligned}$$

Theorem 2.4. Let M be a nonempty subset of R^n and let $f: M \to R$ and $E: R^n \to R^n$ be two functions. If $epi(f) \subset E - e(f)$, and f is convex on $E(M)$. Then *f* is a semi-*E*-convex function on *M*.

Proof. Be similar to the proof of Theorem 2.3.

Theorem 2.5. Let M be a nonempty subset of R^n and let $f: M \to R$ and $E: R^n \to R^n$ be two functions. If $epi(f) \subset E - e(f)$, and $epi_E(f)$ is convex, then *f* is a semi-*E*-convex function on *M*.

Proof. For $x, y \in M$ and $t \in [0,1]$, $(x, f(x)), (y, f(y)) \in epi(f)$

From the condition $epi(f) \subset E - e(f)$, we have $(x, f(x)), (y, f(y)) \in E - e(f)$, thus

$$f(Ex) \le f(x),\ f(Ey) \le f(y),$$

This implies that

$$(Ex, f(x)), (Ey, f(y)) \in epi_E(f).$$

As $epi_E(f)$ is convex, we have

$$((1-t)Ex + tEy, (1-t)f(x) + tf(y))) \in epi_E(f).$$

Then we have

$$\begin{aligned} f((1-t)E(x) + tE(y)) &\le (1-t)f(E(x)) + tf(E(y)) \\ &\le (1-t)f(x) + tf(y). \end{aligned}$$

Definition 2.1. Let $E: R^n \to R^n$ be a mapping, *M* be a nonempty *E*-convex subset of R^n. A function $f: M \to R^+$ is said to a logarithmic E-convex on *E*-convex set M, if $\ln(f(x))$ is E-convex on *E*-convex set M, i.e.

$$f((1-t)E(x) + tE(y)) \le f(Ex)^{1-t} f(Ey)^t, \forall x, y \in M, t \in [0,1]$$

Definition 2.2. Let $E: R^n \to R^n$ be a mapping, *M* be a nonempty *E*-convex subset of R^n. A function $f: M \to R^+$ is said to a logarithmic semi-E-convex on *E*-convex set M, if $\ln(f(x))$ is semi-E-convex on *E*-convex set M, i.e.

$$f((1-t)E(x) + tE(y)) \le f(x)^{1-t} f(y)^t, \forall x, y \in M, t \in [0,1]$$

Theorem 2.6. Let M be a nonempty subset of R^n, for a mapping $E: R^n \to R^n$, and a function $f: R^n \to R$, then f be a logarithmic E-convex on *E*-convex set $M \subset R^n \Rightarrow f$ be E-convex on $M \Rightarrow f$ be quasi-E-convex on M (i.e. $f((1-t)E(x) + tE(y)) \le \max\{f(Ex), f(Ey)\}$).

Proof. For $t \in [0,1]$, and $x, y \in M$, we have

$$\begin{aligned} f((1-t)E(x) + tE(y)) &\le f(Ex)^{1-t} f(Ey)^t \\ &\le (1-t)f(E(x)) + tf(E(y)) \\ &\le \max\{f(Ex), f(Ey)\}. \end{aligned}$$

Theorem 2.7. Let M be a nonempty subset of R^n, $E: R^n \to R^n$ be function, and let $f: R^n \to R$ be logarithmic semi-E-convex on *E*-convex set $M \subset R^n \Rightarrow f$ be semi-E-convex on $M \Rightarrow f$ be quasi-semi-E-convex on M.

Proof. For $t \in [0,1]$, and $x, y \in M$, we have

$$\begin{aligned} f((1-t)E(x) + tE(y)) &\le f(x)^{1-t} f(y)^t \\ &\le (1-t)f(x) + tf(y) \end{aligned}$$

$$\leq \max\{f(x), f(y)\}.$$

Definition 2.3[7]. See Ref. 3 or Ref. 4. Let A and B be two subsets of R^n. We say that A is slack 2-convex with respect to B if, for every $x, y \in A \cap B$ and every $t \in [0,1]$ with the property that $(1-t)x + ty \in B$, we have $(1-t)x + ty \in A$.

The following theorem gives a sufficient condition for f to be a semi-E-convex function using the set $epi^E(f)$.

Theorem 2.8. Let M be a nonempty subset of R^n and let $f : M \to R$ and $E : R^n \to R^n$ be two functions. If M is an E-convex set, $E(M)$ is a convex set, and $f(Ex) \leq f(x)$, $\forall x \in M$ $epi^E(f)$ is a slack 2-convex set with respect to $E(M) \times R$, then f is a semi-E-convex function on M.

Proof. Let $x, y \in M$ and $t \in [0,1]$. Then, $(Ex, f(x))$, $(Ey, f(y)) \in (E(M) \times R) \cap epi^E(f)$.

Since $E(M)$ is a convex set, we have $(1-t)Ex + tEy \in E(M)$; hence,

$$((1-t)Ex + tEy, (1-t)f(x) + tf(y)) \in (E(M) \times R).$$

Since $epi^E(f)$ is a slack 2-convex set with respect to $(E(M) \times R)$, then,

$$((1-t)Ex + tEy, (1-t)f(x) + tf(y)) \in epi^E(f).$$

It follows that there exist $z \in M$, such that, $(1-t)E(x) + tE(y) = E(z)$, and

$$f(z) \leq (1-t)f(x) + tf(y)),$$

Hence, $f((1-t)E(x) + tE(y)) \leq f(z) \leq (1-t)f(x) + tf(y))$

Then, f f is semi-E-convex function on M.

3 Some Results of Semi-E-Convex Programming

Let us consider the following programming problem:

(P) Min $f(x)$,

s.t. $x \in M = \{x \in R^n : g_i(x) \leq 0, i = 1,2,\cdots,m\}$,

where $f : R^n \to R$ and $g_i : R^n \to R$, $i = 1,2,\cdots,m$ are function on R^n.we have the following results.

Theorem 3.1. If $x_0 \in M$ is a fixed point of the mapping $E : R^n \to R^n$ (i.e. $x_0 = Ex_0$), and x_0 is a local minimum of the problem (P) on an E-convex set M, and $f : R^n \to R$ is semi-E-convex on the set M, then x_0 is global minimum of problem (P) on M.

Proof. Let $x_0 \in M$ be a nonglobal minimum of the problem (P) on M, then, there is $y \in M$ such that $f(y) < f(x_0)$, since function $f : R^n \to R$ is semi-E-convex on the set M , and $x_0 \in M$ is a fixed point of the mapping E ,it implies that

$$f((1-t)x_0 + tE(y)) = f((1-t)E(x_0) + tE(y)) \le$$
$$(1-t)f(x_0) + tf(y) < f(x_0),$$

For any small $t \in (0,1)$, which contradicts the local optimality of x_0 for problem (P) .

Hence, x_0 is global minimum of problem (P) on M .

Theorem 3.2. If $x_0 \in M$ is a fixed point of the mapping $E : R^n \to R^n$ (i.e. $x_0 = Ex_0$),and x_0 is a local minimum of the problem (P) on an E-convex set M , and $f : R^n \to R$ is pseudo-semi-E-convex on the set M , then x_0 is global minimum of problem (P) on M .

Proof. Let $x_0 \in M$ be a nonglobal minimum of the problem (P) on M, then, there is $y \in M$ such that $f(y) < f(x_0)$, since function $f : R^n \to R$ is s pseudo-emi-E-convex on the set M and $x_0 \in M$ is a fixed point of the mapping E ,it implies that

$$f(tE(y) + (1-t)x_0) = f(tE(y) + (1-t)E(x_0)) \le$$
$$f(x_0) + t(t-1)b(y, x_0) < f(x_0),$$

For any small $t \in (0,1)$, which contradicts the local optimality of x_0 for problem (P) . Hence, x_0 is global minimum of problem (P) on M .

Theorem 3.3. Assume function $f : R^n \to R$ is a strongly quasi-semi- *E*-convex on a set $M \subset R^n$, then the global optimal solutions of problem (P) is unique.

Proof. Let $x_1, x_2 \in M$ be two different global optimal solutions of problem (P), then, $f(x_1) = f(x_2)$. Since M is E -convex and f is strongly quasi-semi-E-convex, then

$$f((1-t)E(x_1) + tE(x_2)) < \max\{f(x_1), f(x_2)\} = f(x_1),\ \forall t \in (0,1),$$

This contradicts the optimality of x_1 for problem (P). Then, the global optimal solution of the problem (P) is unique.

Theorem 3.4. Let M be a nonempty subset of $R^n, E : R^n \to R^n$ be function, and let $f : R^n \to R$ be pseudo-semi- E -convex on *E*-convex set $M \subset R^n, u \in M$ be fixed point of map E (i.e. $u = Eu$) and

$$\langle f'(Eu), Ev - Eu \rangle \ge 0, \forall v \in M, \tag{1}$$

Then u is a minimum of function *f* on *M*..

Proof. Let $u \in M$ be a non minimum of function f on *M*, then, there is

$v \in M$ such that $f(v) < f(u)$, since f is pseudo-semi-E-convex on E-convex set M, we have

$$f(E(u) + t(E(v) - E(u))) = f(tE(v) + (1 - t)E(u))$$
$$\leq f(u) + t(t - 1)b(v, u) = f(E(u)) + t(t - 1)b(v, u),$$

For all $t \in (0,1)$, and

$$\frac{f(E(u) + t(E(v) - E(u))) - f(E(u))}{t} \leq (t - 1)b(v, u).$$

Letting $t \to 0$, we have

$$\left\langle f'(E(u), E(v) - E(u) \right\rangle \leq -b(v, u) < 0,$$

Which contradicts the condition (1).Hence u is a minimum of function *f* on *M*..

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