REPLACEMENT FIRST AND LAST FOR A PARALLEL SYSTEM WITH CONSTANT AND RANDOM UNITS

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Abstract

This paper observes optimal replacement times for a parallel system with \( n \) units, when it is operating for successive jobs with a random working cycle. The classical approach of "whichever occurs first" and the newly proposed approach of "whichever occurs last" are respectively employed for replacements scheduled at time \( T \) and at working cycle \( Y \), whose policies are called replacement first and replacement last. Two cases when the number of units for this parallel system is a given constant value and a random variable with estimated distribution are considered into modelings. We obtain their expected replacement cost rates and optimal solutions analytically. Further, comparisons of replacement first and replacement last are described in detail to determine which policy could save more replacement cost rate.

1 Introduction

It was shown by graph that the parallel system can operate for a specified mean time by either changing the replacement time or increasing the number of units \([1, 2]\). The reliability measures of many redundant systems \([3]\) and parallel-series systems \([4]\) were computed. Some optimization methods of redundancy allocation for series-parallel systems were studied \([5]\). Recently, a new asymptotic method of computing the MTTF for a large scaled parallel system was proposed \([6]\). Further, optimal number of units and replacement times for a parallel system with a random number of units were considered \([7]\). Several cases of constant and random replacement times for parallel systems were optimized by introducing the shortage and excess costs in scheduling problems \([8]\).

The main purpose of this paper is to observe optimal replacement times for a parallel system with \( n \) units, when it is operating for successive jobs with a random working cycle \([9]\). We propose that when replacement times would be scheduled at a planned time \( T \) and at the end of the random working cycle \( Y \), the newly proposed assumption "whichever occurs last" \([10]\) should be employed into modelings, whose motivations have been explored extensively. The replacement policy with the classical assumption "whichever occurs first" \([1]\) is also used to make comparison of that with "whichever occurs last". We call replacement polices with respective "first" and "last" as replacement first and replacement last in this paper. Further, when the number of units for this parallel system cannot be predefined as a constant value but can be estimated as a random variable \([6, 7]\), the above replacement first and last policies and their comparison are reconsidered analytically.

Throughout the paper, the general assumptions are given as follows:

1) Corrective Replacement: Each unit has an identical failure distribution \( F(t) \) with finite mean \( 1/\lambda \ (0 < \lambda < \infty) \) and the failure rate \( h(t) = f(t)/F(t) \), where \( f(t) \) is a density function of \( F(t) \) and \( \Phi(t) \equiv 1 - \Phi(t) \) for any function \( \Phi(t) \). In addition, \( h(t) \) increases to \( h(\infty) \equiv \lim_{t \to \infty} h(t) \) which might be infinite. The system fails when all of \( n \) units have failed and Corrective Replacement (CR) is done immediately.

2) Preventive Replacement I: The system serves for successive projects with a random cycle \( Y \), whose distribution is \( G(t) \equiv \Pr\{Y \leq t\} \) and mean is \( 1/\theta \equiv \int_0^{\infty} G(t)dt < \infty \). The system will be replaced preventively if it is still working at every end of cycle \( Y \), e.g., replacing the failed units and overhauling the whole system.

3) Preventive Replacement II: However, it is not enough for Preventive Replacement (PR) strategy when only PR I is planned. PR II is supposed to be done at a planned time \( T \ (0 < T < \infty) \) for the high security of system

2 Constant Number \( n \) of Units

It is assumed from common method that the number of units for a parallel system could be predefined as constant numbers such that \( n = 1, 2, \cdots \).
2.1 Replacement First

Suppose that the system is replaced preventively before failure at a planned time $T$ ($0 < T \leq \infty$) or at the end of the working cycle $Y$, whichever occurs first. The policy indicates that replacement should be done as soon as possible before failure when either $T$ or $Y$ is first triggered. Then, the mean time to replacement is

$$
TG(T)[1 - F(T)^n] + \int_0^T t[1 - F(t)^n]dG(t) \\
+ \int_t^T G(t)[1 - F(t)^n]dt = \frac{c_T}{c_F - c_T} \int_0^T G(t)[1 - F(t)^n]dt.
$$

(1)

Therefore, the expected replacement cost rate is

$$
C_F(T) = \frac{c_T + (c_F - c_T) \int_0^T G(t)dF(t)}{\int_0^T G(t)[1 - F(t)^n]dt}.
$$

(2)

where $c_T$ and $c_Y$ are the respective replacement costs at times $T$ and $Y$, $c_F$ is the replacement cost at failure, and $c_F > c_T$ and $c_F > c_R$.

In particular, when $c_T = c_R$, (2) becomes

$$
C_F(T) = \frac{c_T + (c_F - c_T) \int_0^T G(t)dF(t)}{\int_0^T G(t)[1 - F(t)^n]dt}.
$$

(3)

which corresponds to the age replacement with a random replacement when $n = 1$ [2].

We next find an optimal $T_0$ which minimizes $C_F(T)$ in (3). Differentiating $C_F(T)$ with respect to $T$ and setting it equal to zero,

$$
H_n(T) \int_0^T G(t)[1 - F(t)^n]dt - \int_0^T G(t)dF(t)^n = \frac{c_T}{c_F - c_T},
$$

(4)

where

$$
H_n(T) = \frac{nh(t)F(t)^{n-1}}{\sum_{j=1}^{n-1}F(t)^j}.
$$

(5)

It can be easily seen that $H_n(t)$ increases strictly with $t$ to $h(\infty)$ for $n \geq 2$, and decreases strictly with $n$ from $h(t)$ to 0. We denote the left-hand side of (4) be $V_F(T)$, then $V_F(0) = 0$ and

$$
V_F(\infty) = H_n(\infty) \int_0^\infty G(t)[1 - F(t)^n]dt \\
- \int_0^\infty G(t)dt = \frac{dH_n(T)}{dT} \int_0^T G(t)[1 - F(t)^n]dt > 0.
$$

Therefore, we have the following optimal policies:

1. If $V_F(\infty) > c_T/(c_F - c_T)$, then there exists a finite and unique $T_0$ ($0 < T_0 < \infty$) which satisfies (4), and the resulting cost rate is

$$
C_F(T_0) = \frac{c_T}{c_F - c_T} H_n(T_0).
$$

(6)

2. If $V_F(\infty) \leq c_T/(c_F - c_T)$, then $T_0 = \infty$, and the expected cost rate is

$$
C_F(\infty) = \frac{c_T + (c_F - c_T) \int_0^\infty G(t)dF(t)}{\int_0^\infty G(t)[1 - F(t)^n]dt}.
$$

(7)

In particular, when $n \geq 2$ and $G(t) = 1 - e^{-\lambda t}$,

$$
H_n(t) = \frac{n\lambda e^{-\lambda t}(1 - e^{-\lambda t})^{n-1}}{1 - (1 - e^{-\lambda t})^n},
$$

increases strictly with $t$ from 0 to $\lambda$. Thus, if

$$
\lambda \int_0^\infty G(t)[1 - (1 - e^{-\lambda t})^n]dt - \int_0^\infty (1 - e^{-\lambda t})^nG(t)dt > \frac{c_T}{c_F - c_T},
$$

(8)

then a finite $T_0$ exists. In addition, when $G(t) = 1 - e^{-\theta t}$, (8) becomes

$$
\sum_{j=1}^{n-1} \left(n \frac{(n-1)j}{j-1} \frac{\theta + \lambda}{\theta + j \lambda} > \frac{c_F}{c_F - c_T},
$$

(9)

whose left-hand side increases with $n$ to $(\theta + \lambda)/\theta$. So that, if $\lambda/\theta \leq c_T/(c_F - c_T)$ then $T_0 = \infty$ for any $n$.

3 Replacement Last

Suppose that the system is replaced at a planned time $T$ or at the end of the working cycle $Y$, whichever occurs last. It can be understood from this policy that the system is replaced as late as possible when either $T$ or $Y$ is last triggered. Then, the mean time to replacement is

$$
TG(T)[1 - F(T)^n] + \int_T^\infty t[1 - F(t)^n]dG(t) \\
+ \int_0^T tdF(t)^n + \int_T^\infty G(t)dF(t)^n
$$

$$
\int_0^T G(t)[1 - F(t)^n]dt = \int_0^\infty G(t)[1 - F(t)^n]dt.
$$

(10)

Therefore, the expected replacement cost rate is

$$
C_L(T) = \frac{c_F - (c_F - c_T)G(T)[1 - F(T)^n]}{\int_0^T [1 - F(t)^n]dt + \int_T^\infty G(t)[1 - F(t)^n]dt}.
$$

(11)

In particular, when $c_T = c_R$,

$$
C_L(T) = \frac{c_F - (c_F - c_T)G(T)[1 - F(T)^n]}{\int_0^T [1 - F(t)^n]dt + \int_T^\infty G(t)[1 - F(t)^n]dt}.
$$

(12)
Clearly, \( C_L(0) = C_F(\infty) \) and

\[
C_L(\infty) = \int_0^{\infty} [1 - F(t)^n] \, dt, \tag{13}
\]

which is the expected cost rate when the unit is replaced only at failure.

We find an optimal \( T_L^* \) which minimizes \( C_L(T) \) in (12). Differentiating \( C_L(T) \) with respect to \( T \) and setting it equal to zero,

\[
H_n(T) \left\{ \int_0^T [1 - F(t)^n] \, dt + \int_T^{\infty} \overline{G}(t)[1 - F(t)^n] \, dt \right\} - \left\{ 1 - \int_T^{\infty} G(t) \, dt \right\} = \frac{c_T}{c_F - c_T}, \tag{14}
\]

where \( H_n(T) \) is given in (5). Denote the left-hand side of (4) be \( V_F(T) \), then

\[
V_L(0) = H_n(0) \int_0^{\infty} \overline{G}(t)[1 - F(t)^n] \, dt,
\]

\[
\frac{dV_L(T)}{dT} = H_n(T) \left\{ \int_0^T [1 - F(t)^n] \, dt + \int_T^{\infty} \overline{G}(t)[1 - F(t)^n] \, dt \right\} > 0,
\]

\[
V_L(\infty) = h(\infty) \int_0^{\infty} [1 - F(t)^n] \, dt.
\]

Therefore, we have the following optimal policies:

1. If \( V_L(0) \geq c_F/(c_F - c_T) \), then \( T_L^* = 0 \), i.e., the system is replaced only at time \( Y \) and the expected cost rate is given in (7).
2. If \( V_L(0) < c_F/(c_F - c_T) < V_L(\infty) \), then there exists a finite and unique \( T_L^* (0 < T_L^* < \infty) \) which satisfies (14), and the resulting cost rate is

\[
C_L(T_L^*) = \frac{c_F}{c_F - c_T}H_n(T_L^*). \tag{15}\]

3. If \( V_L(\infty) \leq c_F/(c_F - c_T) \), then \( T_L^* = \infty \), i.e., the system is replaced only at failure, and the resulting cost rate is given in (13).

In particular, when \( n \geq 2 \) and \( F(t) = 1 - e^{-\lambda t} \), if

\[
\sum_{j=1}^{n} \frac{1}{j} > \frac{c_F}{c_F - c_T}, \tag{16}
\]

then a finite \( T_L^* (0 < T_L^* < \infty) \) exists.

### 4 Comparison I

We compare the expected cost rates \( C_F(T) \) in (3) and \( C_L(T) \) in (12) for \( n \geq 2 \). From (4) and (14), we denote

\[
L(T) = \int_0^T G(t)[1 - F(t)^n][H_n(T) - H_n(t)] \, dt - \int_T^{\infty} \overline{G}(t)[1 - F(t)^n][H_n(t) - H_n(T)] \, dt. \tag{17}
\]

Clearly,

\[
L(0) = -\int_0^{\infty} \overline{G}(t) \, dt < 0,
\]

\[
L(\infty) = \int_0^{\infty} G(t)[1 - F(t)^n][h(\infty) - H_n(t)] \, dt < 0,
\]

\[
\frac{dL(T)}{dT} = \frac{dH_n(T)}{dT} \left\{ \int_0^T G(t)[1 - F(t)^n] \, dt + \int_T^{\infty} \overline{G}(t)[1 - F(t)^n] \, dt \right\} > 0.
\]

Thus, there exists a finite and unique \( T_P (0 < T_P < \infty) \) which satisfies \( L(T) = 0 \).

Therefore, denoting

\[
V_F(T_P) = H_n(T_P) \int_0^{T_P} \overline{G}(t)[1 - F(t)^n] \, dt - \int_0^{T_P} \overline{G}(t) \, dt,
\]

we have the following comparative results from point of replacement cost rates in (6) and (15):

1. If \( V_F(T_P) \geq c_F/(c_F - c_T) \), then \( T_F^* \leq T_L^* \), i.e., replacement first is better than replacement last.
2. If \( V_F(T_P) < c_F/(c_F - c_T) \), then \( T_F^* > T_L^* \), i.e., replacement last is better than replacement first.

### 5 Random Number \( N \) of Units

Suppose that the number of units for this parallel system cannot be predefined constantly as is done with the traditional method, but it is a random variable \( N \) whose mean can be estimated from historical data [6, 7], which is more suitable for a complex redundancy system with a large or uncertain number of units to complete specified operations. In other words, suppose that \( N \) has a truncated Poisson distribution with parameter \( \beta (0 < \beta < \infty) \),

\[
\Pr\{N = n\} = \frac{1}{1 - e^{-\beta}} \frac{\beta^n}{n!} e^{-\beta} \quad (n = 1, 2, \ldots).
\]

Then, the system has a failure distribution

\[
F_{\beta}(t) = \frac{1}{1 - e^{-\beta}} \sum_{n=1}^{\infty} \frac{\beta^n}{n!} e^{-\beta} F(t)^n = \frac{e^{-\beta T(t)}}{1 - e^{-\beta}}. \tag{19}
\]

It can be seen that when \( \beta \) is large for a complex redundancy system, \( 1 - e^{-\beta} \approx 1 \), then \( F_{\beta}(t) \approx e^{-\beta T(t)} \), which will be used for simplicity in the following discussions.

The expected cost rate of replacement first is, from (3),

\[
C_F(T; \beta) = \frac{c_T}{c_F + c_T} \left[ \int_0^T \overline{G}(t) \beta e^{-\beta T(t)} \, dt \right] \frac{dF(t)}{dT}.
\]

\[
\int_0^T \overline{G}(t)[1 - e^{-\beta T(t)}] \, dt.
\]
Differentiating $C_F(T; \beta)$ with respect to $T$ and setting it equal to zero,
\[
H_\beta(T) \int_0^T G(t)[1 - e^{-\beta F(t)}]dt - \int_T^\infty G(t)[1 - e^{-\beta F(t)}]dF(t) = \frac{c_T}{c_F - c_T},
\]
where
\[
H_\beta(T) \equiv \frac{\beta f(t)e^{-\beta T(t)}}{1 - e^{-\beta T(t)}}.
\]
Thus, if $H_\beta(T)$ increases strictly, then the left-hand side of (20) also increases strictly from 0 to
\[
U_F(\infty) \equiv H_\beta(\infty) \int_0^\infty G(t)[1 - e^{-\beta F(t)}]dF(t) - \int_0^\infty G(t)[1 - e^{-\beta F(t)}]dF(t).
\]
Therefore, if $U_F(\infty) > c_T/(c_F - c_T)$, then there exists a finite and unique $T_P^* (0 < T_P^* < \infty)$ which satisfies (21), and the resulting cost rate is
\[
C_F(T_P^*; \beta) = (c_F - c_T)H_\beta(T_P^*).
\]
Next, the expected cost rate of replacement last is, from (12),
\[
C_L(T; \beta) = \frac{c_F - (c_F - c_T)\int_0^T G(t)[1 - e^{-\beta F(t)}]dF(t)}{\int_0^T [1 - e^{-\beta F(t)}]dt + \int_T^\infty G(t)[1 - e^{-\beta F(t)}]dF(t)}
\]
Differentiating $C_L(T; \beta)$ with respect to $T$ and setting it equal to zero,
\[
H_\beta(T) \left\{ \int_0^T [1 - e^{-\beta F(t)}]dt + \int_T^\infty G(t)[1 - e^{-\beta F(t)}]dF(t) \right\} - \left[1 - \int_T^\infty G(t)[1 - e^{-\beta F(t)}]dF(t) \right] = \frac{c_T}{c_F - c_T},
\]
whose left-hand side increases strictly with $T$, and
\[
U_L(0) = H_\beta(0) \int_0^\infty G(t)[1 - e^{-\beta F(t)}]dF(t)
- \int_0^\infty G(t)[1 - e^{-\beta F(t)}]dF(t)
U_L(\infty) = H_\beta(\infty) \int_0^\infty [1 - e^{-\beta F(t)}]dt - 1.
\]
Therefore, if $U_L(\infty) > c_T/(c_F - c_T)$, then there exists a finite and unique $T_L^* (0 \leq T_L^* < \infty)$ which satisfies (25), and the resulting cost rate is
\[
C_L(T_L^*; \beta) = (c_F - c_T)H_\beta(T_L^*).
\]
\section{Comparison II}
From (21) and (25), we denote
\[
L(T) \equiv \int_0^T G(t)[1 - e^{-\beta F(t)}][H_\beta(T) - H_\beta(t)]dt
- \int_T^\infty G(t)[1 - e^{-\beta F(t)}][H_\beta(t) - H_\beta(T)]dt.
\]
Clearly, $L(T)$ increases strictly with $T$, and
\[
L(0) = -\int_0^\infty G(t)[1 - e^{-\beta F(t)}]dF(t) < 0
L(\infty) = \int_0^\infty G(t)[1 - e^{-\beta F(t)}][h(\infty) - H_\beta(t)]dt.
\]
Thus, there exists a finite and unique $T_P (0 < T_P < \infty)$ which satisfies $L(T) = 0$.
Therefore, denote
\[
U_F(T_P) \equiv H_\beta(T_P) \int_0^{T_P} G(t)[1 - e^{-\beta F(t)}]dF(t)
- \int_0^{T_P} G(t)[1 - e^{-\beta F(t)}]dF(t).
\]
If $U_F(T_P) \geq c_T/(c_F - c_T)$, then $T_P \geq T_L^*$, i.e., the replacement first is better than replacement last, and vice versa.

\section{Conclusions}
We have proposed replacement first and replacement last for a parallel system when it is operating successive jobs with a random working cycle. Obviously, the redundancy system in parallel is required to keep the running reliable. However, the classical “whichever occurs first” in maintenance theory would cause operational interruption during jobs’ working cycle when bivariate replacement candidates are scheduled, such that replacement done at at time $T$ and at working cycle $Y$ proposed in this paper. So that the newly proposed “whichever occurs last” has been employed into modelings to guarantee jobs completion. This paper has also considered replacement first and last when the number of units for this parallel system is a random variable. This case is practical as the system becomes large scaled or when we cannot estimate exactly how many units in this parallel system should be prepared to complete a large amount of jobs. We have compared replacement first and last analytically from the viewpoint of cost rate to find in what cases replacement first or last would save more cost. The paper could be extended and computed easily, and its potential value would be explored in the following studies.

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