OPTIMAL PORTFOLIO SELECTION BASED ON SATISFACTION INDEX

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Abstract

In this paper, optimal portfolio selection with uncertain returns is studied, and corresponding model based on the satisfaction index is proposed. In the model, the risk is taken as the sum of the absolute deviation of the risky assets in stead of covariance, the transaction cost is taken as the v-shaped function of the difference between the existing and new portfolio. An efficient way is given to transform a non-linear problem into a linear problem, which alleviate the computational difficulty greatly. Numerical result showed that the proposed method is capable of helping investor to find efficient portfolios according to his/her preference.

1 Introduction

Fluctuation in stock market is unpredictable and it is random in nature. This is a difficult task to achieve without planning and evaluating investment alternatives. The portfolio must incorporate what the investor believes to be an acceptable balance between risk and reward. Markowitz mean-variance model of portfolio selection [12, 13] is one of the best known models in finance and unanimously recognized to contribute in the development of modern portfolio theory. It explores how risk-averse investors can construct optimal portfolio assets taking into consideration the trade-off between expected returns and market risk.

Portfolio selection issue has served as a basis for the development of modern financial theory over the past five decades and continuously gained an interest among scholars [10], [1], [18], [6]. However, contrary to its theoretical reputation, it is not used extensively to construct large-scale portfolios. One of the most important reasons for this is the computational difficulty associated with solving a large-scale quadratic programming problem with a dense covariance matrix. The modern portfolio theory then evolved to Capital Asset Pricing Theory [21] when risk free rate asset was included into the portfolio and then evolved to Arbitrage Pricing Theory in which the computation was largely reduced. Konno and Yamazaki [10] used the absolute deviation risk function to replace the risk function in Markowitz’s model and formulated a mean absolute deviation portfolio optimization model. It turns out that the mean absolute deviation model maintains the favorable properties of Markowitz’s model and removes most of the principal difficulties in solving Markowitz’s model. Simaan [22] provided a thorough comparison of the mean variance model and the mean absolute deviation model. Speranza [23] used the semi-absolute deviation to measure the risk and formulated a portfolio selection model. Furthermore, the Markowitz model is too basic from practical point of view and ignores many constraints faced by real-world investors: trading limitations, size of portfolio, transaction costs, etc ([1], [2], [19], [24]). Investment strategies may be theoretically very profitable before taking into account transaction costs and taxation issue, but the situation can become worse (such as inefficient portfolio) and completely different when these last constraints are incorporated. Any realistic investment portfolio selection must support transaction costs among other practical limitations.


Traditionally, it has been assumed that the distribution functions of possibility returns are known while solving portfolio selection models. However, new securities and classes of assets have emerged in recent times and it is not always possible for an investor to specify them. In some cases, for instance, historical data of stocks are not available. In such cases, the uncertain returns of assets may be determined as interval numbers by using experts’ knowledge.

The paper is organized as follows. In Section 2, we firstly give some notations for interval numbers and
briefly introduce some interval arithmetics and then an order of relations over intervals is introduced. Based on this concept, an approach to compare interval numbers is proposed. Lastly, an interval absolute deviation model for portfolio selection is proposed. In Section 3, an example is given to illustrate our approach. In Section 4, some concluding remarks are also given.

2 Methods

2.1 Preliminary

An interval number can be viewed as a special fuzzy number whose membership function takes value 1 over the interval, and 0 anywhere else. For an interval number \(a = [a_-, a_+]\), the median \(m(a)\) and width \(w(a)\) is defined by \(m(a) = (a_- + a_+)/2\) and \(w(a) = (a_+ - a_-)/2\), respectively. The operations on intervals used in this paper are as follows:

Definition 2.1: For any two interval numbers \(a = [a_-, a_+]\) and \(b = [b_-, b_+]\), operations of intervals can be defined as

\[
a + b = [a_- + b_-, a_+ + b_+], \quad a - b = [a_- - b_+, a_+ - b_-],
\]

\[
a \pm k = [a_- \pm k, a_+ \pm k],
\]

\[
ka = k[a_-, a_+] = \left\{ \begin{array}{ll} [ka_-, ka_+] & k \geq 0 \\ [ka_+, ka_-] & k < 0 \end{array} \right.,
\]

where \(k\) is a real number.

The three operations of intervals are equivalent to the operations of addition, subtraction and scalar multiplication of fuzzy numbers via the extension principle.

[8] suggested an order relation between two intervals as follows:

Definition 2.2: For any two interval numbers \(a = [a_-, a_+]\) and \(b = [b_-, b_+]\), the order relation between a and b is defined as

\[
a \leq b \quad \text{if and only if} \quad a_- \leq b_- \quad \text{and} \quad a_+ \leq b_+,
\]

\[
a < b \quad \text{if and only if} \quad a_- \leq b_- \quad \text{and} \quad a_+ < b_+.
\]

For describing the interval inequality relation in detail, the following concept was introduced in [7].

Definition 2.3: For any two interval numbers \(a = [a_-, a_+]\) and \(b = [b_-, b_+]\), there is an interval inequality relation \(a \prec b\) between the two interval numbers \(a\) and \(b\) if and only if \(m(a) \leq m(b)\). Furthermore, if \(\bar{a} \leq \bar{b}\), we say the interval inequality relation \(a \prec b\) between \(a\) and \(b\) is optimistic satisfactory; if \(\bar{a} > \bar{b}\), we say the interval inequality relation \(a \prec b\) between \(a\) and \(b\) is pessimistic satisfactory. Finally, the pessimistic and optimistic satisfaction index of the interval inequality relation \(a \prec b\) can be defined as

\[
PSD(a \prec b) = 1 + \frac{b - \bar{a}}{w(a) + w(b)}
\]

and

\[
OSD(a \prec b) = \frac{b - \bar{a}}{w(a) + w(b)}
\]

respectively.

Since \(b > a\) implies that there may be some possibility for \(b\) to be greater than \(a\), the definition 2.2 and 2.3 don’t contain all possibilities for interval inequality to hold. Therefore, the following concept combined interval inequality relation and satisfaction index is introduced.

Definition 2.4: For any two interval numbers \(a = [a_-, a_+]\) and \(b = [b_-, b_+]\), there is an interval inequality relation \(a \preceq b\) between the two interval numbers \(a\) and \(b\) if and only if \(a \leq b\). The satisfaction index of the interval inequality relation \(a \preceq b\) is defined as

\[
SD(a \preceq b) = \max\left\{ \frac{2(b - \bar{a})}{w(a) + w(b)}, 0 \right\}.
\]

According to definitions of the pessimistic and the optimistic satisfaction indices, we can see that the range of the pessimistic satisfaction index can be \([0, 1]\), and the range of the optimistic satisfaction can be \([0, \infty]\). Our proposed satisfaction index is more general concept than those of [7].

2.2 Portfolio selection model under linear programming

Assume that an investor wants to allocate his wealth among \(n\) risky assets offering random rates of returns and chooses \(x_i\), the proportion invested in asset \(i\), \(1 \leq i \leq n\) for \(n\) assets. The constraints are \(\sum_{i=1}^{n} x_i = 1\) and \(x_i \geq 0, i = 1, 2, \ldots, n\). The return \(r_{it}\) for the \(ith\) asset, \(1 \leq i \leq n\), is a random variable, with arithmetic mean return \(r_{ai} = \frac{1}{T} \sum_{t=1}^{T} r_{it}\), where \(r_{it}\) can be determined by historical data. Let \(x = (x_1, x_2, \ldots, x_n)^T\) and \(r = (r_1, r_2, \ldots, r_n)^T\). In this paper the transition cost for the \(ith\) asset \(c_i\) employs \(v\)-shape function, that is

\[
c(x_i) = k_i|x_i - x_i^0|, \quad i = 1, 2, \ldots, n.
\]

where \(x_i^0 = (x_1^0, x_2^0, \ldots, x_n^0)^T\) is a given assets owned by the investor and \(k_i \geq 0\) the transition cost for the unit of \(ith\) asset. So the total transition cost of the portfolio \(x\) is described

\[
c(x) = \sum_{i=1}^{n} c(x_i) = \sum_{i=1}^{n} k_i|x_i - x_i^0|.
\]

The uncertain expected return of the risky asset \(j = 1, \ldots, n\) can be represented as the following interval number:

\[
\bar{r}_i = [r_{ij}^L, r_{ij}^U] = [\min\{r_{adj}, r_{hj}\}, \max\{r_{adj}, r_{hj}\}],
\]

where \(r_{hj}\) is the historical return tendency factor of risky asset \(j\) and \(r_{adj}\) is the arithmetic mean factor of risky asset \(j\). So the expected return interval of portfolio \(x = (x_1, x_2, \ldots, x_n)^T\) in the future can be represented as

\[
\bar{r}(x) = \sum_{i=1}^{n} \bar{r}_i x_i.
\]
After subtracting the transaction costs, the net expected return interval of portfolio \(x = (x_1, x_2, \cdots, x_n)^T\) in the future can be represented as

\[
\hat{r}(x) = \hat{r}(x) - c(x) = \left\{ \sum_{i=1}^{n} \hat{r}_i x_j - \sum_{i=1}^{n} k_i |x_i - x_i^0| \right\}.
\]

Because the expected returns on securities are considered as interval numbers, we may consider the semi-absolute deviation of the rates of return of portfolio \(x\) below the expected return over all the past periods as an interval number too. Since the expected return interval of portfolio \(x = (x_1, x_2, \cdots, x_n)^T\) is

\[
\hat{r}(x) = \left\{ \sum_{i=1}^{n} \hat{r}_i x_j, \sum_{i=1}^{n} \hat{r}_j x_i \right\}.
\] (4)

We can get the semi-absolute deviation interval of return of portfolio \(x = (x_1, x_2, \cdots, x_n)^T\) below the expected return over the past period \(t, t = 1, \cdots, T\). It can be represented as

\[
\tilde{w}_t(x) = [\underline{w}_t(x), \overline{w}_t(x)],
\]

where

\[
\underline{w}_t(x) = \max\left\{ \sum_{i=1}^{n} (\hat{r}_j - r_{ij}) x_j, 0 \right\}
\]

and

\[
\overline{w}_t(x) = \max\left\{ \sum_{i=1}^{n} (\hat{r}_i - r_{ij}) x_i, 0 \right\}.
\]

Then the average value of the semi-absolute deviation interval of return of portfolio \(x\) below the uncertain expected return over all the past period, can be represented as

\[
\tilde{w}(x) = \frac{1}{T} \sum_{t=1}^{T} \tilde{w}_t(x) = [\underline{w}(x), \overline{w}(x)].
\] (6)

We use \(\tilde{w}(x)\) to measure the risk of portfolio \(x\). Suppose that the investor wants to maximize the return of a portfolio after subtracting the transaction costs within some given level of risk. If the risk tolerance interval \(\tilde{w} = [\underline{w}, \overline{w}]\) is given, the mathematical formulation of the portfolio selection problem is

\[
\begin{align*}
\max \hat{r}(x) = & \sum_{i=1}^{n} \hat{r}_i x_j - \sum_{i=1}^{n} k_i |x_i - x_i^0| \\
\text{s.t.} & \sum_{i=1}^{n} x_i = 1, \\
& 0 \leq x_i \leq u_i, i = 1, \cdots, n.
\end{align*}
\] (7)

where \(\underline{w}\) represents the pessimistic tolerated risk level, and \(\overline{w}\) represents the optimistic tolerated risk level. (7) is an optimization problem with interval coefficients and, therefore, techniques of classical linear programming can not be applied unless the above interval optimization problem is reduced to a standard linear programming structure. In the following, we perform this conversion.

We introduce the order relation in the interval function of (7). Based on the concept of satisfaction index proposed above, the interval inequality relation \(\tilde{w}(x) \preceq [\underline{w}, \overline{w}]\) in (7) is expressed by a crisp inequality as follows:

\[
SD(\tilde{w}(x) \preceq [\underline{w}, \overline{w}]) \geq \alpha,
\]

where satisfaction index \(\alpha \in [0, \infty)\) is given by the investor.

Thus the interval linear programming problem (7) can be represented by interval linear programming problem in which the objective function is interval number and the constraint conditions are crisp equality and inequalities. The interval objective function linear programming problem is represented as follows:

\[
\begin{align*}
\max \hat{r}(x) = & \sum_{i=1}^{n} \hat{r}_i x_j - \sum_{i=1}^{n} k_i |x_i - x_i^0| \\
\text{s.t.} & \sum_{i=1}^{n} x_i = 1, \\
& 0 \leq x_i \leq u_i, i = 1, \cdots, n.
\end{align*}
\] (8)

The satisfactory solution of (8) is equivalent to the non-inferior solution set of the following bi-objective programming problem for given satisfaction index \(\alpha \in [0, \infty)\).

\[
\begin{align*}
\max \hat{r}(x) = & \sum_{i=1}^{n} \hat{r}_i x_j - \sum_{i=1}^{n} k_i |x_i - x_i^0| \\
\text{s.t.} & \sum_{i=1}^{n} x_i = 1, \\
& 0 \leq x_i \leq u_i, i = 1, \cdots, n.
\end{align*}
\] (9)

By the multi-objective programming theory, the non-inferior solution to (9) can be generated by solving the following linear programming problem:

\[
\begin{align*}
\max & \sum_{i=1}^{n} (\lambda \hat{r}_j + (1 - \lambda) r_{ij}) x_j - \sum_{i=1}^{n} k_i |x_i - x_i^0| \\
\text{s.t.} & \sum_{i=1}^{n} x_i = 1, \\
& 0 \leq x_i \leq u_i, i = 1, \cdots, n.
\end{align*}
\] (10)

Thus, (10) may be rewritten as follows by introducing the concrete form of \(SD(\tilde{w}(x) \preceq [\underline{w}, \overline{w}])\):

\[
\begin{align*}
\max & \sum_{i=1}^{n} (\lambda \hat{r}_j + (1 - \lambda) r_{ij}) x_j - \sum_{i=1}^{n} k_i |x_i - x_i^0| \\
\text{s.t.} & \sum_{i=1}^{n} x_i = 1, \\
& (1 - \alpha) |\underline{w}(x) - \overline{w}| + \alpha |\overline{w}(x) - \underline{w}| \leq 0.
\end{align*}
\] (11)
To solve (11), we consider the following transformation.

First, we introduce a new variable $x_0$ such that

$$\sum_{i=0}^{n} k_i |x_i - x_i^0| \leq x_0.$$ 

Let

$$d^+_i = \frac{|x_i - x_i^0| + (x_i - x_i^0)}{2},$$

$$d^-_i = \frac{|x_i - x_i^0| - (x_i - x_i^0)}{2},$$

$$y^+_i = \frac{\sum_{i=1}^{n} (r_j - r_{i,j}) x_j + \sum_{i=1}^{n} (r_j - r_{i,j}) x_j}{2},$$

$$y^-_i = \frac{\sum_{i=1}^{n} (r_j - r_{i,j}) x_j + \sum_{i=1}^{n} (r_j - r_{i,j}) x_j}{2},$$

and

$$y = \frac{1}{T} \sum_{i=1}^{T} y^+_i, \quad \bar{y} = \frac{1}{T} \sum_{i=1}^{T} y^-_i.$$

Then, (11) is equivalent to the following standard linear programming problem.

$$\begin{align*}
\max \ & \sum_{i=1}^{n} (\lambda x_j + (1-\lambda) y_j) x_j - \sum_{i=1}^{n} k_i |x_i - x_i^0| \\
& \quad (1-\alpha) |y - \bar{y}| + \alpha |y - \bar{y}| \leq 0, \\
& \quad \sum_{i=1}^{n} k_i (d^+_i - d^-_i) \leq x_0, \\
& \quad y^+_i - \sum_{i=1}^{n} (r_j - r_{i,j}) x_j \geq 0, \\
& \quad y^-_i - \sum_{i=1}^{n} (r_j - r_{i,j}) x_j \geq 0, \\
& \quad d^+_i - d^-_i = x_i - x_i^0, \\
& \quad d^+_i \geq 0, d^-_i \geq 0, \\
& \quad y^+_i \geq 0, y^-_i \geq 0, \\
& \quad \sum_{i=1}^{n} x_i = 1, \\
& \quad 0 \leq x_i \leq u_i, i = 1, \ldots, n.
\end{align*}$$

One can use several algorithms of linear programming to solve (12) efficiently, for example, the simplex method. So we can solve the original portfolio selection problem (7) by solving (12).

### 3 Results

In this section, we suppose that an investor chooses 6 componental stocks and we collected historical data of the 6 stocks during 5 periods, using one month as a period. The historical return tendency $r_{hj}$ was obtained by $r_{hj} = \frac{1}{n} \sum_{t=1}^{n} r_{tj}$, where $n$ is amount of the most recent periods (we took $n = 3$). The expected rate of return intervals are given in Table 1.

#### Table 1: The expected rates of returns intervals ($10^{-4}$).

<table>
<thead>
<tr>
<th>Stock</th>
<th>Lower</th>
<th>Upper</th>
</tr>
</thead>
<tbody>
<tr>
<td>St1</td>
<td>838</td>
<td>1000</td>
</tr>
<tr>
<td>St2</td>
<td>562</td>
<td>989</td>
</tr>
<tr>
<td>St3</td>
<td>220</td>
<td>513</td>
</tr>
<tr>
<td>St4</td>
<td>600</td>
<td>760</td>
</tr>
<tr>
<td>St5</td>
<td>450</td>
<td>1040</td>
</tr>
<tr>
<td>St6</td>
<td>488</td>
<td>780</td>
</tr>
</tbody>
</table>

Suppose the investor stipulates risk level interval $\tilde{w} = [0.015, 0.035]$. By the method proposed in the above section, we can solve the portfolio selection problem by solving (12). For the given risk level interval $\tilde{w}$, more satisfactory portfolios can be generated by varying the values of the parameters $\lambda$ and $\alpha$ in (12).

As an example to illustrate, we fix the parameter $\alpha = 0.5$ and vary the parameter $\lambda$. The return intervals, the risk intervals and the values of parameters of portfolios are listed in Table 2. The corresponding portfolios are listed in Table 3.

#### Table 2: The return intervals, the risk intervals and the value of parameters of portfolios

<table>
<thead>
<tr>
<th>Portfolio</th>
<th>$\alpha$</th>
<th>$\lambda$</th>
<th>Return Interval</th>
<th>Risk Interval</th>
</tr>
</thead>
<tbody>
<tr>
<td>Portfolio 1</td>
<td>0</td>
<td>0.015</td>
<td>[0.015, 0.033]</td>
<td>[0.0603, 0.0946]</td>
</tr>
<tr>
<td>Portfolio 2</td>
<td>0.15</td>
<td>0.015</td>
<td>[0.015, 0.033]</td>
<td>[0.0603, 0.0946]</td>
</tr>
<tr>
<td>Portfolio 3</td>
<td>0.30</td>
<td>0.015</td>
<td>[0.015, 0.030]</td>
<td>[0.0683, 0.0931]</td>
</tr>
<tr>
<td>Portfolio 4</td>
<td>0.45</td>
<td>0.015</td>
<td>[0.015, 0.030]</td>
<td>[0.0684, 0.0919]</td>
</tr>
<tr>
<td>Portfolio 5</td>
<td>0.60</td>
<td>0.015</td>
<td>[0.015, 0.030]</td>
<td>[0.0710, 0.0918]</td>
</tr>
<tr>
<td>Portfolio 6</td>
<td>0.75</td>
<td>0.015</td>
<td>[0.015, 0.030]</td>
<td>[0.0710, 0.0917]</td>
</tr>
<tr>
<td>Portfolio 7</td>
<td>0.90</td>
<td>0.015</td>
<td>[0.015, 0.030]</td>
<td>[0.0710, 0.0917]</td>
</tr>
</tbody>
</table>

The investor may choose his own investment strategy from the portfolios according to his attitude towards the securities’ expected returns and the degree of portfolio risk with which he is comfortable. If the investor is not satisfied with any of these portfolios, he may obtain more by solving the parametric linear programming problems (12) for other values of parameter $\lambda$ and $\alpha$.

### 4 Conclusion

In this paper, by introducing a concept of inclusive satisfaction index of the interval inequality relation, an approach to compare interval numbers is given. By using the approach, the interval semi-absolute deviation model can be converted into a parametric linear programming problem with two parameters. We represented the interval inequality by the satisfaction index inequality unlike equality of [7]. One can find a satisfactory solution to the original problem by solving the corresponding parametric linear programming problem. An investor may choose a satisfactory investment strategy according to an optimistic or pessimistic attitude by choosing proper values of the parameter $\lambda$ and $\alpha$. The model can help the investor to find an efficient portfolio according to his/her preference.
Table 3: The allocations of portfolio for $\alpha = 0.5$.

<table>
<thead>
<tr>
<th>Portfolio</th>
<th>Allocation 1</th>
<th>Allocation 2</th>
<th>Allocation 3</th>
<th>Allocation 4</th>
<th>Allocation 5</th>
<th>Allocation 6</th>
<th>Allocation 7</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda$</td>
<td>0</td>
<td>0.15</td>
<td>0.30</td>
<td>0.45</td>
<td>0.60</td>
<td>0.75</td>
<td>0.90</td>
</tr>
<tr>
<td>stock1</td>
<td>0.2378</td>
<td>0.2378</td>
<td>0.4462</td>
<td>0.5550</td>
<td>0.5639</td>
<td>0.5716</td>
<td>0.5716</td>
</tr>
<tr>
<td>stock2</td>
<td>0.4317</td>
<td>0.4317</td>
<td>0.3424</td>
<td>0.2433</td>
<td>0.2343</td>
<td>0.2235</td>
<td>0.2235</td>
</tr>
<tr>
<td>stock3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.0364</td>
<td>0.0402</td>
<td>0.0451</td>
<td>0.0451</td>
</tr>
<tr>
<td>stock4</td>
<td>0.0831</td>
<td>0.0831</td>
<td>0.1518</td>
<td>0.1608</td>
<td>0.1617</td>
<td>0.1598</td>
<td>0.1598</td>
</tr>
<tr>
<td>stock5</td>
<td>0.2474</td>
<td>0.2474</td>
<td>0.0597</td>
<td>0.0046</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>stock6</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

References


