INVERSE DEA MODEL WITH CONSIDERING
RETURNS TO SCALE AND ELASTICITY

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Keywords: Data Envelopment Analysis(DEA), Efficiency Score, Returns to Scale(RTS), Most Productive Scale Size(MPSS), Elasticity

Abstract
This paper discusses a new kind of inverse data envelopment analysis(DEA) model with considering returns to scale and elasticity of decision making unit(DMU). An inverse DEA model can be used for a DMU to estimate its input/output levels when some or all of its input/output entities are revised. Different from original inverse DEA model, the new model allows the efficiency score being changed, which is more of economical background. Under this hypothesis, by finding most production scale size of the observed DMU, we propose a function to compute the approximate change of efficiency score, and then we construct an algorithm to solve the new inverse DEA model. Numerical example is discussed at last.

1 Introduction
Data envelopment analysis was firstly proposed to evaluate relative efficiency for not-for-profit organizations[1], as variety of new DEA applications and research being raised, new developments in performance forecasting and resource estimation come into important research topics. Zhang and Cui[2] developed a project evaluation system which firstly extend DEA model to inverse DEA field. They proposed two kinds of resource allocation problem: resource allocation problem and investment analysis problem, the former one was modeled to a one-dimensional parameter problem. Wei and Zhang et,al.[3] extended the above two problems to a more general field characterized by inverse DEA problem, which extends the concept of the inverse optimization problem to the DEA context. Inverse DEA method provides a new tool available to management scientists for performance analysis. In [3], the inverse DEA problem is transformed into and solved as a multi-objective programming problem. It is also shown that in some special cases, the inverse DEA problem can be simplified as a single-objective linear programming problem. In [4], a common algorithm is provided to solve the inverse DEA problem.

In the above research, inverse DEA model was built under the assumption that the DMU’s efficiency score is unchanged. But when we consider the economical concept such as returns to scale and elasticity in inverse DEA model, we find that the assumption which restricts the efficiency score unchanged is not always valid. In economical concept of elasticity, if the DMU’s scale is changed, its efficiency is changed according to its returns to scale. In this paper, we pose an example analysis to show a different kind of inverse DEA model taking into account the returns to scale and elasticity, and pose an approximate algorithm to forecast the changes of the DMU’s efficiency score.

2 Background
2.1 DEA models
Charnes, Cooper and Rhodes firstly introduced the C-CR ratio definition which generalized the single-output to single-input classical engineering-science ratio definition to multiple outputs and inputs without requiring preassigned weights. Since the very beginning of DEA studies, various extension of the CCR model have been proposed, among which the BCC model is representative. The BCC model has its production possibility set and production frontiers different with CCR model’s. CCR and BCC model is presented as follows(see [5]).

(CCR) \[ \min \theta \]
\[ \text{s.t.} \]
\[ X\lambda \leq \theta x_o \]
\[ Y\lambda \geq y_o \]
\[ \lambda \geq 0 \]

(BCC) \[ \min \theta \]
\[ \text{s.t.} \]
\[ X\lambda \leq \theta x_o \]
\[ Y\lambda \geq y_o \]
\[ e\lambda = 1 \]
\[ \lambda \geq 0 \]

The CCR model is built on the assumption of constant returns to scale of activities, but BCC model take into account of various returns to scale, and categories various DMU into four types: increasing, constant, decreasing returns to scale and the congestion.
2.2 Inverse DEA

An investment analysis problem in [2] is abstracted as follows: a set of DMUs have efficiency indices \( \theta_1, \ldots, \theta_n \). Assign an increment, \( \Delta x^i \geq 0 \), to the input of \( S_i \) which has efficiency score \( \theta_i \). Find the "largest" additional resources, \( \Delta y^j \), to the output of \( S_i \) such that the resulted status of \( S_i \) remains its efficiency score unchanged. The IPM (Investment Prediction Model) is to find the "largest" solution \( \Delta y^j \) such that the optimal value \( \theta_{n+1} = \theta_i \), where \( \theta_i \) is given by (CCR), by defining an additional DMU \( S_{n+1} \) with input and output vectors \( (x^i + \Delta x^i, y^j + \Delta y^j) \), where \( \Delta y \) is an unknown vector variable. Moreover, if there is a "price making" for the inputs, in other words, there exists a set of weights \( W = (W_1, \ldots, W_m)^T \) associated with the \( m \) inputs, then the (IPM) model is solved by a linear programming problem (IPM*).

\[
\text{(IPM*)} \quad \begin{align*}
\text{Min} & \quad \theta \equiv \theta_{n+1} \\
\text{s.t.} & \quad \sum_{k=1}^{n} \lambda_k W^T x^k + \lambda_{n+1} W^T (x^i + \Delta x^i) \\
& \quad \leq \theta W^T (x^i + \Delta x^i) \\
& \quad \sum_{k=1}^{n} \lambda_k y^k + \lambda_{n+1} (y^j + \Delta y^j) \\
& \quad \geq y^j + \Delta y^j \\
& \quad \lambda_k \geq 0, \ k = 1, \ldots, n + 1.
\end{align*}
\]

Cui et.al. proposed an algorithm for solving the above model, see [4]. It is equally to the following multi-objective programming (MOP).

\[
\text{(MOP)} \quad \begin{align*}
\text{max} & \quad (\Delta y_{1o}, \ldots, \Delta y_{so}) \\
\text{s.t.} & \quad X \lambda \leq \theta_i (x_o + \Delta x) \\
& \quad Y \lambda \geq y_o + \Delta y \\
& \quad \lambda \geq 0
\end{align*}
\]

2.3 MPSS and scale elasticity

By using (CCR) model, we can define the most productive scale size of DMU, by the following formula.

\[
\text{(MPSS)} \quad \tilde{x}_{io} = \frac{\theta_i x_{io} - s_1^{-*}}{\sum_{j=1}^{n} \lambda_j^{-*}}, \quad \tilde{y}_{io} = \frac{y_{ro} - s_1^{+*}}{\sum_{j=1}^{n} \lambda_j^{+*}}
\]

By using the concept of MPSS, we can initiate with the following expression \( (x_o, y_o, \alpha, \beta) \), in which the vector \( (\alpha, \beta) \) are scalars representing expansion or contraction factors according to whether \( \alpha, \beta > 1 \) or \( \alpha, \beta < 1 \) are applied to the inputs and outputs[5].

\[
\text{(Scale)} \quad \begin{align*}
\text{max} & \quad \beta / \alpha \\
\text{s.t.} & \quad \beta y_o \leq \sum_{j=1}^{n} y_j \lambda_j
\end{align*}
\]

Suppose \( (\alpha^*, \beta^*) \) is the optimal solution to model (Scale), then \( (\tilde{x}_{io}, \tilde{y}_{io}) = (\alpha^* x_o, \beta^* y_o) \) corresponding to the MPSS point of given DMU. Then we have the following form:

\[
\begin{align*}
\alpha^* &= \frac{y_{ro} - \tilde{y}_{ro} / \tilde{x}_{io} - \tilde{x}_{io}}{x_{ro} / y_{ro} - \Delta x_{ro} / \Delta y_{ro}} \\
\beta^* &= \frac{y_{ro} - \tilde{y}_{ro} / \tilde{x}_{io} - \tilde{x}_{io}}{x_{ro} / y_{ro} - \Delta x_{ro} / \Delta y_{ro}}
\end{align*}
\]

We can use the above formulations to generalize the elasticity concept to the case of multiple-inputs and multiple-outputs by noting that:

\[
\begin{align*}
\alpha^* &= \frac{y_{ro} - \tilde{y}_{ro} / \tilde{x}_{io} - \tilde{x}_{io}}{x_{ro} / y_{ro} - \Delta x_{ro} / \Delta y_{ro}} \\
\beta^* &= \frac{y_{ro} - \tilde{y}_{ro} / \tilde{x}_{io} - \tilde{x}_{io}}{x_{ro} / y_{ro} - \Delta x_{ro} / \Delta y_{ro}}
\end{align*}
\]

For each of these mix pairs we then have a measure of scale change associated with movements to the region of MPSS.

3 Reanalyzing the inverse DEA problem

Although the inverse DEA method is of many great metrics, there is a limitation of it in performance forecasting. Because it is for short-term forecasting, it does not take into consideration the changing case of DMU’s efficiency score[3]. When some additional inputs are allocated to the inefficient DMU, inverse DEA method constructs model to get the output changes with supposing that the DMU maintains its efficiency score. Actually, when considering returns to scale and elasticity concept, we can find that the DMU’s efficiency is changed as its input scale changed, so we reconsider the inverse DEA method in the meaning of returns to scale or elasticity and propose a new kind of inverse DEA method.

3.1 Elasticity shows that \( \Delta \theta \neq 0 \)

Example 1: Three DMUs with single input and single output are considered as follows.

<table>
<thead>
<tr>
<th>Table 1: Example 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>DMU</td>
</tr>
<tr>
<td>input</td>
</tr>
<tr>
<td>output</td>
</tr>
</tbody>
</table>

By using the (BCC) and (MPSS) model discussed in section 2, we can identify that A’s returns to scale is increasing and A’s most production scale size is same with B. The input distance of A between real scale and
the most productive scale size is 2. Considering the resource is not redundant, we assume that $0 \leq \Delta x \leq 2$, as if $\Delta x > 2$, the A’s returns to scale may be deceasing kind, using same method can get the answer. But in economics, the increment inputs to a DMU whose returns to scale is decreasing is always not a much smart investment, so in this paper, we only considering the increasing returns to scale case.

By using inverse DEA model, when decision maker allocate $\Delta x = 0.2$ to DMU A, assume that A’s efficient score unchanged, it is easily turn out that A moves to $A''(1.2,1.2)$ as shown in fig2. Since the model (Elasticity) reflects the proportional change in output relative to the proportional change in input. Hence when input increases from $x = 1$ to $x = 1.2$, the proportional increase is $(1.2 - 1)/1 = 0.2$, and for A the elasticity $(1 - \beta^*)/(1 - \alpha^*) = 2$ means that the proportional increase in output will be $2 \times 0.2 = 0.4$, which is the proportional output change moving from A to $A''(1.2,1.4)$ as shown in fig2. Then we can get $\theta_A = \theta_{A'} = 0.75 < \theta_{A''} = 0.875$.

3.2 Forecasting the $\Delta \theta$

We can see from the figure 1 that when increase $\Delta x = 0.2$ to A, we can find that the point A may reach to point $A''$, instead of point $A''$ which is solved by the original inverse DEA method. We forecast the $\Delta \theta$ by a function $\Delta \theta = g(\Delta x)$. Because the exact function $g(\bullet)$ is hard to get, we pose a linear approximate function to denote it, which is $\Delta \theta = \rho \Delta x$. So our main task changes to find the most satisfying $\rho$. As in example 1, we compute $\rho$ by using the reference pair $(\Delta \bar{x}, \Delta \bar{\theta})$, where $\Delta \bar{x}$ is the x-coordinate distance between point A and point B (A’s MPSS), and $\Delta \bar{\theta} = \theta_B - \theta_A$. So we can get $\rho = \Delta \theta / \Delta \bar{x} = (1 - 0.75)/0.5 = 0.5$, so $\Delta \theta = \rho \Delta x = 0.1$. If increase 0.2 units of input to A, the efficiency score of A is changed to $\theta_A = 0.85$.

By using the inverse DEA model, combined with the new $\theta$ calculated above, we can forecast the output changes of A that $\Delta y = 0.36$. Obviously, $\Delta y/\Delta x = 1.8 > \frac{\Delta \theta}{\Delta x}$, which denotes that DMU_A is increasing returns to scale.

By analyzing example 1, we have an intuitive understanding of the strategy to solve the new kind of inverse DEA model by considering returns to scale of the DMU observed. But example 1 has a limitation that its DMU has single input and single output, when it is extended to multiple inputs and multiple output-s case, we can also obtain the most productive scale size of each DMU, the main difficulty is in the stage of forecasting $\Delta \theta$. As can be known from the example 1, we forecast the DMU_A’s $\Delta \theta$ by using the function $\Delta \theta = \rho \Delta x$. When it is extended to multiple inputs case, the $\Delta x$ is a m-dimensional vector. If we can project it to one-dimension by a function that makes sense in DEA method, we can get a solution to the general case.

3.3 Algorithm for the new model

In this section, we construct an algorithm to solve the new inverse DEA model by using the common weights[7]. In paper [8], the authors proposed a
most compromising common weights based on DEA method. Suppose the common weights is $cw = (u_1, \ldots, u_s; v_1, \ldots, v_m)$, the algorithm for the new inverse DEA model is as follows, where the observed DMU$_o$ is with multiple inputs and multiple outputs, and the decision maker wants to allocate extra inputs $\Delta x$ to DMU$_o$:

Algorithm 1: Algorithm for the new inverse DEA model

Step 1: compute the efficiency score $\theta_o$ of DMU$_o$ by using (CCR) model;

Step 2: determine the returns to scale of DMU$_o$ by using (BCC) model. W.l.o.g., we assume that the DMU$_o$’s returns to scale is increasing;

Step 3: calculate the most productive scale size $(\hat{x}_o, \hat{y}_o)$ of DMU$_o$ by using (MPSS) model;

Step 4: compute ratio $\rho$ by using the reference pair $(\Delta \hat{x}, \Delta \hat{y})$, where $\Delta \hat{y} = 1 - \theta_o$, $\Delta \hat{x} = \sum_{i=1}^m \hat{v}_i x_{oi} - \sum_{m=1}^m v_i x_{io}$, and the ratio is $\rho = \Delta \hat{y} / \Delta \hat{x}$;

Step 5: compute $\Delta \hat{y} = \sum_{i=1}^n \hat{v}_i \Delta x_i$, we assume that $\Delta \hat{y} \leq \Delta \hat{x}$. So $\Delta \hat{y} = \rho \Delta \hat{x}$;

Step 6: $\theta_o = \theta_o + \Delta \hat{y}$. Bring the new parameters to model (MOP), to obtain $\Delta y$.

The advantage of the above algorithm is that it improves the original inverse DEA model by considering the situation that $\theta$ is changed, and it extends the new inverse DEA model from single-input case to multi-input case. The limitation of the algorithm is that the approximation of $\Delta \theta$ by using function $\Delta \theta = \rho \Delta x$ is not always close to the economical practice because the function only uses the efficiency score but ignores the elasticity of the observed DMU.

4 Numerical example

Example 2: 15 supermarkets are investigated, which has staff number, scale as input index and service, benefits as output index. If the boss wants to allocate more inputs $\Delta x = (4, 5)$ to the DMU O, how much output increments can DMU$_o$ get?

By using the original inverse DEA method which maintains the efficiency score of DMU$_o$, we can get the output equals to $\hat{y} = (13.08, 16.31)$. By using algorithm 1, we compute the $\Delta \theta = 0.06$, and get the maximal output $y = y_o + \Delta y = (15.06, 18.79)$, which is obviously greater than $\hat{y}$. The new solution reflects the potential improvements of DMU$_o$ can get when put more resource on it.

5 Conclusion

Inverse DEA method brings new flavor to the theory and application field of DEA method. This paper propose a new inverse DEA model and its related algorithm which improve the original inverse DEA method. The idea is obtained from economics disciplines and the method is constructed based on some important results in DEA fields such as common weights analysis, returns to scale analysis, elasticity analysis and resource allocation methods based on DEA. This study, combined with inverse DEA, extend the research and application range of DEA method.

References


[8] Xiaoya Li, and Jinchuan Cui, Extra resource allocation problem with the most compromise common weights based on DEA method, Lecture notes in Operational Research, ORSC and APORC, 2007, pp. 397-404.