COMMUNICATION LEADING TO COALITION
NASH EQUILIBRIUM II
– S4N-KNOWLEDGE CASE –

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Abstract

In this paper the new concept of coalition Nash equilibrium of a strategic game is introduced, and it is shown that a communication among the players in a coalition leads to the equilibrium through messages. A coalition Nash equilibrium for a strategic game consists of (1) a subset $S$ of players, (2) independent mixed strategies for each member of $S$, (3) the conjecture of the actions for the other players not in $S$ with the condition that each member of $S$ maximises his/her expected payoff according to the product of all mixed strategies for $S$ and the other players' conjecture. However, this paper stands on the Bayesian point of view as follows: The players start with the same prior distribution on a state-space. In addition they have private information which is given by a reflexive and transitive binary relation on the state space. Each player in a coalition $S$ predicts the other players' actions as the posterior of the others' actions given his/her information. He/she communicates privately their beliefs about the other players' actions through messages among all members in $S$ according to the communication network in $S$, which message is information about his/her individual conjecture about the others' actions. The recipients update their belief by the messages. Precisely, at every stage each player communicates privately not only his/her belief about the others' actions but also his/her rationality as messages according to a protocol and then the recipient updates their private information and revises her/his prediction. In this circumstance, we show that the conjectures of the players in a coalition $S$ regarding the future beliefs converge in the long run communication, which lead to a coalition Nash equilibrium for the strategic game.

1 Introduction

Recently, researchers in economics, AI, and computer science become entertained lively concerns about relationships between knowledge and actions. At what point does an economic agent sufficiently know to stop gathering information and make decisions? There are also concerns about cooperation and knowledge. What is the role of sharing knowledge to making cooperation among agents.

Considering a coalition among agents, we tacitly understand that each agent in the coalition share their individual information and so they commonly know each other. In mathematical point of view yet a little is known what structure they have to know commonly. The aim of this paper is to fill the gap. Our point is that in a coalition, the members does not necessary have common-knowledge to each others but they communicate his/her own beliefs on the others to each other through messages.

The purposes of this paper are to introduce the concept of coalition Nash equilibrium of a strategic game, and to show that a communication among the players in a coalition leads to the equilibrium through messages. A coalition Nash equilibrium for a strategic game consists of (1) a subset $S$ of players, (2) independent mixed strategies for each member of $S$, (3) the conjecture of the actions for the other players not in $S$ with the condition that each member of $S$ maximizes his/her expected payoff according to the product of all mixed strategies for $S$ and the other players' conjecture.

This paper analyses the solution concept from the Bayesian point of view: The players start with the same prior distribution on a state-space. In addition they have private information which is given by a partition of the state space. Each player in a coalition $S$ predicts the other players' actions as the posterior of the others' actions given his/her information. He/she communicates privately their beliefs about the other players' actions through messages among all members in $S$ according to the communication network in $S$, which message is information about his/her individual conjecture about the others' actions. The recipients update their belief by the messages. Precisely,
at every stage each player communicates privately not only his/her belief about the others’ actions but also his/her rationality as messages according to a protocol and then the recipient updates his private information and revises his/her prediction.

In this circumstance, we shall show that

**Main theorem.** Suppose that the players in a strategic form game have the knowledge structure associated with a partial information with a common prior distribution. In a communication process of the game according to a protocol with revisions of their beliefs about the other players’ actions, the profile of their future predictions converges to a coalition Nash equilibrium of the game in the long run.

This paper is organized as follows. Section 2 recalls the knowledge structure associated with a reflexive and transitive information structure, and we extend a game on knowledge structure. The communication process for the game is introduced where the players send messages about their conjectures about the other players’ action. In Section 3 we give the formal statement of the main theorem (Theorem 1) and will illustrate it by a simple example. The proof will be given in the sequel section 4. Finally we conclude with remarks.

## 2 The Model

Let \( \Omega \) be a non-empty finite set called a state-space, \( N \) a set of finitely many players \( \{1, 2, \ldots, n\} \) at least two \((n \geq 2)\), and let \( 2^\Omega \) be the family of all subsets of \( \Omega \). Each member of \( 2^\Omega \) is called an event and each element of \( \Omega \) is called a state. Let \( \mu \) be a probability measure on \( \Omega \) which is common for all players. For simplicity it is assumed that \( (\Omega, \mu) \) is a finite probability space with \( \mu \) full support.\(^1\)

### 2.1 Information and Knowledge\(^2\)

A \( RT \)-information structure\(^3\) \( (\Omega, (\Pi_i)_{i \in N}) \) consists of a state space \( \Omega \) and a class of the mappings \( \Pi_i \) of \( \Omega \) into \( 2^\Omega \) such that

\[
\begin{align*}
\text{Ref} & \quad \omega \in \Pi_i(\omega); \\
\text{Trn} & \quad \xi \in \Pi_i(\omega) \implies \Pi_i(\xi) \subseteq \Pi_i(\omega);
\end{align*}
\]

The structure is the Kripke semantics for the logic \( S4n \).

Given our interpretation, the set \( \Pi_i(\omega) \) will be interpreted as the set of all the states of nature that \( i \) knows to be possible at \( \omega \), and we will therefore call \( \Pi_i \) \( i \)’s possibility operator on \( \Omega \) and also will call \( \Pi_i(\omega) \) \( i \)’s information set at \( \omega \). Further, an player \( i \) for whom \( \Pi_i(\omega) \subseteq \{E\} \) knows, in the state \( \omega \), that some state in the event \( E \) has occurred. In this case we say that in the state \( \omega \) the player \( i \) knows \( E \).

### Definition 1. The \( S4n \)-knowledge structure

\[
(\Omega, (\Pi_i)_{i \in N}, (K_i)_{i \in N})
\]

consists of a partitioned information structure \( (\Omega, (\Pi_i)_{i \in N}) \) and a class of \( i \)’s \( S4n \)-knowledge operator \( K_i \) on \( 2^\Omega \) such that \( K_i E \) is the set of states of \( \Omega \) in which \( i \) knows that \( E \) has occurred; that is, \( K_i E = \{ \omega \in \Omega \mid \Pi_i(\omega) \subseteq E \} \).

The set \( K_i E \) will be interpreted as the set of states of nature for which \( i \) knows \( E \) to be possible.

We record the properties of \( i \)’s knowledge operator\(^4\):

- For every \( E, F \) of \( 2^\Omega \),
  \[ K_i \ (E \cap F) = K_i E \cap K_i F; \]
  \[ K_i F \subseteq F; \]
  \[ 4 \quad K_i F \subseteq K_i K_i F; \]

**Remark 1.** \( i \)’s possibility operator \( \Pi_i \) is uniquely determined by \( i \)’s knowledge operator \( K_i \) satisfying the above four properties: For \( \Pi_i(\omega) = \cap_{\omega \in K_i E} E \).

**Remark 2.** The RT-information structure is a partial structural form game have the knowledge structure associated

\[ \begin{align*}
\text{Sym} & \quad \text{If } \xi \in \Pi_i(\omega) \text{ then } \omega \in \Pi_i(\xi). \\
\text{Ref} & \quad \text{If } \omega \in \Pi_i(\omega) \text{ then } \omega \in \Pi_i(\xi).
\end{align*} \]

The structure is the Kripke semantics for the logic \( S5n \), and the postulate is equivalent to

\[
\begin{align*}
5 \quad & \Omega \setminus K_i E \subseteq K_i(\Omega \setminus K_i E),
\end{align*}
\]

### 2.2 Game on knowledge structure\(^5\)

By a game \( G \) we mean a finite strategic form game \( (N, (A_i)_{i \in N}, (g_i)_{i \in N}) \) with the following structure and interpretations: \( N \) is a finite set of players \( \{1, 2, \ldots, i, \ldots, n\} \) with \( n \geq 2 \), \( A_i \) is a finite set of \( i \)’s actions (or \( i \)’s pure strategies) and \( g_i \) is an \( i \)’s payoff function of \( A_i \) into \( R \), where \( A \) denotes the product \( A_1 \times A_2 \times \cdots \times A_n \), \( A_{-i} \) the product \( A_1 \times A_2 \times \cdots \times A_{i-1} \times A_{i+1} \times \cdots \times A_n \). We denote by \( g \) the \( n \)-tuple \( (g_1, g_2, \ldots, g_n) \) and by \( a_{-i} \) the \((n-1)\)-tuple \( (a_1, a_{-1}, a_{i+1}, \ldots, a_n) \) for \( a \) of \( A \). Furthermore we denote \( a_{-i} = (a_i)_{i \in N} \) for each \( i \in N \). A probability distribution \( \sigma_i \) on \( A_i \) is called an \( i \)’s mixed strategy for a game \( G \). We denote by \( \Delta(A_i) \) the set of all \( i \)’s mixed strategies, so we will denote \( \Delta(A) = \prod_{i=1}^{n} \Delta(A_i) \) and \( \Delta(A_j) = \prod_{i \neq j} \Delta(A_i) \).

### Definition 2. A profile \((\sigma_i)_{i \in N} \) of mixed strategies is called a Nash equilibrium if for each \( i \in N \) and for every \( b_i \in A_i \), we have

\[
\sum_{a_{-i} \in A_{-i}} g_i(a_i, a_{-i}) \prod_{j \in N \setminus \{i\}} \sigma_j(a_j) \geq \sum_{a_{-i} \in A_{-i}} g_i(b_i, a_{-i}) \prod_{j \in N \setminus \{i\}} \sigma_j(a_j)
\]

\(^{1}\)That is, \( \mu(\omega) \neq 0 \) for every \( \omega \in \Omega \).

\(^{2}\)C.f.; Bacharach [3], Binmore [4] for the information structure and the knowledge operator.

\(^{3}\)RT-information stands for reflexive and transitive information.

\(^{4}\)According to these we can say the structure \((\Omega, (K_i)_{i \in N}) \) is a model for the multi-modal logic \( S5n \).

\(^{5}\)C.f., Aumann and Brandenburger [2].
A probability distribution $\varphi_i \in \Delta(A_\omega) = \Delta(A_i \cap \omega)$ is said to be $i$'s overall conjecture (or simply $i$'s conjecture). For each player $j$ other than $i$, this induces the marginal distribution on $j$'s actions, we call it $i$'s individual conjecture about $j$ (or simply $i$'s conjecture about $j$). Functions on $\mathcal{Q}$ are viewed like random variables in the probability space $(\Omega, \mu)$. If $x$ is a such function and $x$ is a value of it, we denote by $[x = x]$ (or simply by $[x]$) the set $\{\omega \in \Omega | x(\omega) = x\}$.

The information structure $(\Pi_i)$ with a common prior $\mu$ yields the distribution on $A \times \Omega$ defined by $q_i(a, \omega) = \mu(\{a = a\} \cap \Pi_i(\omega))$; and the $i$'s overall conjecture defined by the marginal distribution $q_i(a_\omega) = \mu(\{a_\omega = a_\omega\} \cap \Pi_i(\omega))$ which is viewed as a random variable of $\varphi_i$. We denote by $[\varphi_i = \varphi_i]$ the intersection $\bigcap_{a_\omega \in A_\omega} [q_i(a_\omega) = \varphi_i(a_\omega)]$ and denote by $[\varphi]$ the intersection $\bigcap_{\omega \in \Omega} [q_i(\omega) = \varphi_i(\omega)]$. Let $g_i$ be a random variable of $i$'s payoff function $g_i$ and $a_i$ a random variable of $i$'s action $a_i$.

According to the Bayesian decision theoretical point of view, we assume that each player is aware of his own actions; i.e., letting $a_i = a_i \subseteq [a_i]$ and for every action $a_i$ of $A_i$, $i$'s action $a_i$ is said to be actual at a state $\omega$ if $\omega \in [a_i = a_i]$; and the payoff off actions $g_i = (g_i, g_i, \ldots, g_i)$ is said to be actually played at a state $\omega$ if $\omega \in [g_i] := \bigcap_{\omega \in \Omega} [g_i(\omega) = g_i(\omega)]$. Let $\exp$ denote the expectation defined by

$$\text{Exp}(g(b_i, a_i); \omega) := \sum_{a_i \in A_i} g_i(b_i, a_i) q_i(a_i, \omega).$$

By a coalition $S$ we mean $S$ is a non-empty subset of $N$. Let $(\sigma_i)_{i \in S}$ be a profile of mixed strategies of $G$ for a coalition $S$. By $S$-expectation of $i$'s pay off function $g_i$ at $\omega$ we mean

$$\text{Exp}_S(g_i(a_i, a_i); \omega) := \sum_{a_i \in A_i} \text{Exp}(g_i(a_i, a_i); \omega).$$

Definition 3. A profile $(\sigma_i)_{i \in S}$ is called a coalition $S$-Nash equilibrium of $G$ if each member $i$ in $S$ maximizes his/her $\text{Exp}_S(g_i(a_i, a_i); \omega)$ for every $\omega \in \Omega$; i.e., $\text{Exp}_S(g_i(a_i, a_i); \omega) \geq \text{Exp}_S(g_i(a_i, a_i); \omega)$ for every $b_i$ in $A_i$.

A coalition $S$ is said to be rational at $\omega$ if for every $i \in S$, each $i$'s actual action $a_i$ maximizes the expectation of his actually played pay off function $g_i$ at $\omega$ when the other players actions are distributed according to his conjecture $q_i(\cdot | \sigma_{-i}, \omega)$. Formally, letting $g_i = g_i(\omega)$ and $a_i = a_i(\omega)$, $\text{Exp}(g_i(a_i, a_i); \omega) \geq \text{Exp}(g_i(b_i, a_i); \omega)$ for every $b_i$ in $A_i$. Let $R_i$ denote the set of all of the states at which $i$ is rational.

2.3 Protocol

We assume that the players communicate by sending messages. Let $T$ be the time horizontal line $\{0, 1, 2, \ldots, t, \ldots\}$. A protocol on a coalition $S$ of a game $G$ is a mapping $\mathcal{P}_{S} : T \times S \times S \times (S(t), r(t))$ such that $s(t) \neq r(t)$. Here $t$ stands for time and $s(t)$ and $r(t)$ are, respectively, the sender and the recipient of the communication which takes place at time $t$. Simply we call it a $S$-protocol. We consider the protocol as the directed graph whose vertices are the set of all members in $S$ and such that there is an edge (or an arc) from $i$ to $j$ if and only if there are infinitely many $t$ such that $s(t) = i$ and $r(t) = j$.

A protocol $\mathcal{P}_{S}$ is said to be fair if the graph is strongly-connected; in words, every player in this protocol communicates directly or indirectly with every other player infinitely often. It is said to contain a cycle if there are players $i_1, i_2, \ldots, i_\ell$ with $\ell \geq 3$ such that for all $m < k$, $i_m$ communicates directly with $i_{m+1}$, and such that $i_\ell$ communicates directly with $i_1$. The communications is assumed to proceed in rounds.

2.4 Communication on coalition

Let $S$ be a coalition of $G$. A coalition $S$-communication process $\pi_S(G)$ with revisions of players’ conjectures $(\varphi_i(\cdot, \omega))_{i \in S}$ according to a protocol for a game $G$ is a tuple

$$\pi_S(G) = (G, (\Omega, \mu), \mathcal{P}_S, (\Pi_i)_{i \in S}, (K^i)_{i \in S}, (\varphi_i(\cdot, \omega))_{i \in S})$$

with the following structures: the players have a common prior $\mu$ over $\Omega$, the protocol $\mathcal{P}_S$ among $N$, $\mathcal{P}_S(t) = (s(t), r(t))$, is fair and it satisfies the conditions that $r(t) = s(t + 1)$ for every $t$ and that the communications proceed in rounds. The revised information structure $\Pi_i$ at time $t$ is the mapping of $\Omega$ into $\mathcal{P}^t$ for player $i \in S$. If $i = s(t)$ is a sender at $t$, the message sent by $i$ to $j = r(t)$ is $M_j$. An $s$-tuple $(\varphi_i(\cdot, \omega))_{i \in S}$ is a revision process of individual conjectures. These structures are inductively defined as follows:

- Set $\Pi_0(\omega) = (\sigma_0(\omega)$.
- Assume that $\Pi_i(\omega)$ is defined. It yields the distribution $q_i(\cdot, \omega) = \mu(\{a = a\} \cap \Pi_i(\omega)$ Whence

$R_i^t$ denotes the set of all the state $\omega$ at which $i$ is rational according to his conjecture $q_i(\cdot, \omega)$; that is, $i$'s actual action $a_i$ maximizes the expectation of his pay off function $g_i$ being actually played at $\omega$ when the other players actions are distributed according to his conjecture $q_i(\cdot, \omega)$ at time $t$.

6Cf.: Parikh and Krasucki [14]
7There exists a time $m$ such that for all $t$, $Pr_S(t) = Pr_S(t + m)$. The period of the protocol is the minimal number of all $m$ such that for every $t$, $Pr_S(t + m) = Pr_S(t)$.
8Formally, letting $g_i = g_i(\omega)$, $a_i = a_i(\omega)$, the expectation at time $t$, $Ex^t_i$, is defined by $Ex^t_i(g_i(a_i, a_i); \omega) := (a_i, a_i, \cdot, \cdot, \cdot, \cdot)$. An $i$ player in $S$ is said to be $S$-rational according to his conjecture $q_i(\cdot, \omega)$ at $\omega$ if for all $b_i$ in $A_i$, $Ex^t_i(g_i(a_i, a_i); \omega) \geq Ex^t_i(g_i(b_i, a_i); \omega)$.
Theorem 1. Suppose that the players in a strategic form game $G$ have the $\mathbf{S}4\mathbf{m}$-knowledge structure with $\mu$ a common prior. Let $S$ be a coalition in a game $G$. In the coalition $S$-communication process $\pi_S(G)$ according to a protocol $\mathcal{P}_S$ among all members in $S$, the $|S|$-tuple of their conjectures $(\varphi_i^{t_i})_{(i,t_i)\in S\times T}$ converges to a coalition $S$-Nash equilibrium of the game in finitely many rounds.

Remark 4. When $S$ is the ground coalition $N$, the above theorem shows that the conjectures of the players leads to a Nash equilibrium through communication. This is shown in Matsuhisa [7].

Remark 5. The notion of common-knowledge control to form a Nash equilibrium for a game (Aumann and Brandenburger [21]), but it cannot do to form a coalition Nash equilibrium.

The below example will show the situation:

Example 1. Let us consider the three persons game $G = (N, (A_i)_{i \in N}, (g_i)_{i \in N})$ as follows;

- The set of players $N = \{1, 2, 3\}$;
- The action sets $A_1 = \{H, T\}$, $A_2 = \{H, T\}$, $A_3 = \{W, E\}$.

The pay-off functions $g_1, g_2, g_3$ are given by the table 1:

<table>
<thead>
<tr>
<th></th>
<th>H</th>
<th>W</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g_1$</td>
<td>0.1</td>
<td>0.2</td>
<td>0.1</td>
</tr>
<tr>
<td>$g_2$</td>
<td>0.1</td>
<td>0.2</td>
<td>0.1</td>
</tr>
<tr>
<td>$g_3$</td>
<td>0.1</td>
<td>0.2</td>
<td>0.1</td>
</tr>
</tbody>
</table>

Table 1: The game $G$ has the unique Nash equilibrium $(\frac{1}{2}H + \frac{1}{2}T, \frac{1}{2}H + \frac{1}{2}T, W)$.

We let start the situation: Each player knows his/her own actions, but he/she cannot know the other players’ action. To model the situation we introduce the game $G$ as a Bayesian game equipped with the below information partition $(\Pi_i)_{i=1,2,3}$:

- The state space $\Omega = \{\omega_1, \omega_2, \ldots, \omega_8\}$;
- $\mu$ is the equal probability measure on $2^\Omega$; i.e., $\mu(\omega) = \frac{1}{8}$;
- The partitions $(\Pi_i)_{i=1,2,3}$ on $\Omega$:
  - The partition $\Pi_1$ on $\Omega$:
    \[
    \Pi_1(\omega) = \{\omega_1, \omega_2, \omega_5, \omega_6\} \\
    (\omega = \omega_1, \omega_2, \omega_5, \omega_6);
    \]
    \[
    \Pi_1(\omega) = \{\omega_3, \omega_4, \omega_7, \omega_8\} \\
    (\omega = \omega_3, \omega_4, \omega_7, \omega_8).
    \]
  - The partition $\Pi_2$ on $\Omega$:
    \[
    \Pi_2(\omega) = \{\omega_1, \omega_3, \omega_5, \omega_7\} \\
    (\omega = \omega_1, \omega_3, \omega_5, \omega_7);
    \]
    \[
    \Pi_2(\omega) = \{\omega_2, \omega_4, \omega_6, \omega_8\} \\
    (\omega = \omega_2, \omega_4, \omega_6, \omega_8).
    \]
  - The partition $\Pi_3$ on $\Omega$:
    \[
    \Pi_3(\omega) = \{\omega_1, \omega_2, \omega_3, \omega_4\} \\
    (\omega = \omega_1, \omega_2, \omega_3, \omega_4);
    \]
    \[
    \Pi_3(\omega) = \{\omega_5, \omega_6, \omega_7, \omega_8\} \\
    (\omega = \omega_5, \omega_6, \omega_7, \omega_8).
    \]
• $a_i$ is defined by

$$
a_1(\omega) = H (\omega = \omega_1, \omega_2, \omega_5, \omega_6);
$$

$$
a_1(\omega) = T (\omega = \omega_3, \omega_4, \omega_7, \omega_8);
$$

$$
a_2(\omega) = h (\omega = \omega_1, \omega_3, \omega_5, \omega_7);
$$

$$
a_2(\omega) = t (\omega = \omega_2, \omega_4, \omega_6, \omega_8);
$$

$$
a_3(\omega) = W (\omega = \omega_1, \omega_2, \omega_3, \omega_4);
$$

$$
a_3(\omega) = E (\omega = \omega_5, \omega_6, \omega_7, \omega_8).
$$

We can observe that the conjectures

$$\phi_1(a_2) = q_1(a_1; \omega)$$

at $\omega_5$ are

- $\phi_2(a_1) = \phi_3(a_1) = \frac{1}{2}H + \frac{1}{2}T$
- $\phi_1(a_2) = \phi_2(a_2) = \frac{1}{2}h + \frac{1}{2}t$
- $\phi_0(a_3) = \phi_2(a_3) = \frac{1}{2}W + \frac{1}{2}E.$

This shows that for each player $i$, any other players than $i$ must agree on every $i'$ actions, but these distributions $(\phi_2(a_1), \phi_1(a_2), \phi_2(a_3))$ cannot form the Nash equilibrium for $G$, but $(\phi_2(a_1), \phi_1(a_2),) = (\phi_3(a_1), \phi_1(a_3))$ is a Nash equilibrium. It should be noted that $(\phi_2(a_1), \phi_1(a_3))$ is commonly known among (1,3), and so the notion of common-knowledge cannot always yield a coalition Nash equilibrium.

We consider the coalition (1,3)-communication process for game $G$ equipped with the protocol $Pr : T \rightarrow S = \{1,3\}$. After 2 rounds communication, the below information partition can be obtained: \(\Pi_1, \Pi_2, \Pi_3\):

- The partitions \(\Pi_{1,2,3}\) on $\Omega$:
  - The partition $\Pi_1$ on $\Omega$:
    $$
    \Pi_1^\omega = \begin{cases}
    (\omega_1,\omega_2) & (\omega = \omega_1, \omega_2); \\
    (\omega_3,\omega_4) & (\omega = \omega_3, \omega_4); \\
    (\omega_5,\omega_6) & (\omega = \omega_5, \omega_6); \\
    (\omega_7,\omega_8) & (\omega = \omega_7, \omega_8).
    \end{cases}
    $$

- The partition $\Pi_3$ on $\Omega$ is the same as the initial partition $\Pi_3$:

- The partition $\Pi_{1,2,3}$ on $\Omega$ is the same as the initial partition $\Pi_{1,2,3}$:

We can observe that the conjectures $\phi_{1,2}(a_2) = q_{1,2}(a_3; \omega)$ at $\omega_5$ are

- $\phi_{2,3}(a_1) = \phi_{5,6}(a_1) = \frac{1}{2}H + \frac{1}{2}T$
- $\phi_{1,3}(a_2) = \phi_{5,6}(a_2) = \frac{1}{2}h + \frac{1}{2}t$
- $\phi_{1,2}(a_3) = \phi_{2,3}(a_3) = W,$

and these distributions $(\phi_{1,2}(a_1), \phi_{1,2}(a_2), \phi_{1,2}(a_3))$ is a (1,3)-Nash equilibrium. Furthermore, $(\phi_{1,2}(a_1), \phi_{1,2}(a_2), \phi_{1,2}(a_3))$ forms the Nash equilibrium for $G$.

4 Proof of Theorem

The proof of Theorem 1 is based on the below proposition:

**Proposition 1.** Notation and assumptions are the same in Theorem 1. For any members $i, j \in S$, their conjectures $q_{1,2}^\omega$ and $q_{1,2}^\omega$ on $A \times \Omega$ must coincide; that is, $q_{1,2}^\omega(a; \omega) = q_{1,2}^\omega(a; \omega)$ for every $a \in A$ and $\omega \in \Omega$.

**Proof.** On noting that $P_{FS}$ is fair, it suffices to verify that $q_{1,2}^\omega(a; \omega) = q_{1,2}^\omega(a; \omega)$ for $(i, j) = (s(\infty), r(\infty))$. Since $\Pi_i(\omega) \subseteq [a]_i$ for all $\omega \in [a]_i$, we can observe that $q_{1,2}^\omega(a_i; \omega) = q_{1,2}^\omega(a_i; \omega)$, and we let define the partitions $\{W_{1,2}^\omega(\omega) \mid \omega \in \Omega\}$ and $\{Q_{1,2}^\omega(\omega) \mid \omega \in \Omega\}$ of $\Omega$ as follows:

$$W_{1,2}^\omega(\omega) = \bigcap_{a \in A} [q_{1,2}^\omega(a_i, \omega) = q_{1,2}^\omega(a_i, \omega)],$$

$$Q_{1,2}^\omega(\omega) = [\xi \in \Omega \mid \Pi_{1,2}^\omega(\omega) \cap W_{1,2}^\omega(\omega) \cap Q_{1,2}^\omega(\omega) \cap W_{1,2}^\omega(\omega)].$$

It follows that

$$Q_{1,2}^\omega(\omega) \subseteq W_{1,2}^\omega(\omega) \quad \text{for all } \xi \in W_{1,2}^\omega(\omega),$$

and hence $W_{1,2}^\omega(\omega)$ can be decomposed into a disjoint union of components $Q_{1,2}^\omega(\xi)$ for $\xi \in W_{1,2}^\omega(\omega)$.

$$W_{1,2}^\omega(\omega) = \bigcup_{k=1,2,3} Q_{1,2}^\omega(\xi_k) \text{ for } \xi_k \in W_{1,2}^\omega(\omega).$$

It can be observed that

$$\mu([a = a] W_{1,2}^\omega(\omega)) = \sum_{k=1}^m \lambda_k \mu([a = a] Q_{1,2}^\omega(\xi_k)) \quad \text{(1)}$$

for some $\lambda_k > 0$ with $\sum_{k=1}^m \lambda_k = 1$.

On noting that $W_{1,2}^\omega(\omega)$ is decomposed into a disjoint union of components $Q_{1,2}^\omega(\xi)$ for $\xi \in W_{1,2}^\omega(\omega)$, it can be observed that

$$q_{1,2}^\omega(a_i; \omega) = \mu([a = a] W_{1,2}^\omega(\omega)) = \mu([a = a] Q_{1,2}^\omega(\xi_k)) \quad \text{(2)}$$

for any $\xi_k \in W_{1,2}^\omega(\omega)$. Furthermore, we can verify that for every $\omega \in \Omega$,

$$\mu([a = a] W_{1,2}^\omega(\omega)) = \mu([a = a] Q_{1,2}^\omega(\omega)).$$

In fact, we first note that $W_{1,2}^\omega(\omega)$ can also be decomposed into a disjoint union of components $Q_{1,2}^\omega(\xi)$ for $\xi \in W_{1,2}^\omega(\omega)$. We shall show that for every $\xi \in W_{1,2}^\omega(\omega)$,

$$\mu([a = a] W_{1,2}^\omega(\omega)) = \mu([a = a] Q_{1,2}^\omega(\xi)).$$

For: Suppose

\[11\] This property is called the *convexity* for the conditional probability $\mu(X|\omega)$ in Parikh and Krasucki [14].

\[12\] This equation follows immediately from Fundamental lemma proved first in Matsuhisa [7].
not, the disjoint union $G$ of all the components $Q_j(\xi)$ such that $\mu([a = a]) W^\infty(\omega) = \mu([a = a]) Q^\infty(\xi)$ is a proper subset of $W^\infty(\omega)$. It can be shown that for some $\omega_0 \in W^\infty(\omega) \setminus G$ such that $Q_j(\omega_0) = W^\infty(\omega) \setminus G$. On noting that $\mu([a = a]) G = \mu([a = a]) W^\infty(\omega)$ it follows immediately that $\mu([a = a]) Q^\infty(\omega_0) = \mu([a = a]) W^\infty(\omega)$, in contradiction. Now suppose that for every $\omega_0 \in W^\infty(\omega \setminus (G \cup Q^\infty(\omega_0) \cup Q^\infty(\omega_1) \cup \cdots \cup Q^\infty(\omega_k)))$ in contradiction also, because $\Omega$ is finite.

In viewing (1), (2) and (3) it follows that

$$q^\infty_\xi(a; \omega) = \sum_{k=1}^{m} \lambda_k q^\infty_{\xi_k}(a; \xi_k)$$

for some $\xi_k \in W^\infty(\omega)$. Let $\xi_\omega$ be the state in $\{\xi_1, \ldots, \xi_m\}$ that attains the maximal value of all $q^\infty_{\xi_k}(a; \xi_k)$ for $k = 1, 2, 3, \ldots, m$, and let $\xi_\omega \in \{\xi_1, \ldots, \xi_m\}$ be the state that attains the minimal value. By (4) we obtain that $q^\infty_\xi(a; \zeta_\omega) \leq q^\infty_\xi(a; \omega) \leq q^\infty_\xi(a; \xi_\omega)$ for $(i, j) = (s(\infty), \infty)$. (4)

On continuing this process according to the fair protocol $P_{SS}$, it can be plainly verified that: For each $\omega \in \Omega$ and for any $t \geq 1$,

$$q^\infty_\xi(a; \zeta) \leq \cdots \leq q^\infty_\xi(a; \omega) \leq q^\infty_\xi(a; \omega)$$

for some $\zeta_1, \ldots, \zeta_t, \zeta_\omega, \xi_\omega \in \Omega$, and thus $q^\infty_\xi(a; \omega)$ because $q^\infty_\xi(a; \omega) \leq q^\infty_\xi(a; \xi_\omega)$ and $q^\infty_\xi(a; \xi_\omega) = q^\infty_\xi(a; \xi)$ for every $\zeta, \xi \in \Omega$. In completing the proof.

**Proof of Theorem 1:** We denote by $\Gamma(i)$ the set of all the players who directly receive the message from $i \in S$; i.e., $\Gamma(i) = \{ j \in S | (i, j) = P_{SS}(i) \text{ for some } t \in T \}$. Let $F_j$ denote $[\varphi^\infty_{j, i}] := \cap_{a_i \in A_i} \{ q^\infty_{j, i}(\omega) = \varphi^\infty_{j, i}(\omega) \}$. It is noted that $F_j \cap F_{j'} = \emptyset$ for $i, j \in S$. We observe the first point that for each $i \in N, j \in \Gamma(i)$ and for every $a, \mu([a = a] F_i \cap F_j) = \varphi^\infty_{j, i}(a)$. Then summing over $a$, we can observe that $\mu([a = a] F_i \cap F_j) = \varphi^\infty_{j, i}(a)$ for $a \in A_i$. View of Proposition 1 it can be observed that $\varphi^\infty_{j, i}(a)$ is independent of the choices of every $j \in S$ other than $i$. We set the probability distribution $\sigma_i$ on $A_i$ by $\sigma_i(a) := \varphi^\infty_{j, i}(a)$, and set the profile $\sigma_S = (\sigma_i)_{i \in S}$.

We observe the second point that for every $a \in \Pi_{i \in S} \text{Supp}(\sigma_i)$,

$$\varphi^\infty_{j, i}(a) = \prod_{j \in S} \sigma_j(a_j) \mu([a = a] \cap S_j(\omega)) :$$

In fact, noting the protocol is fair, we can take the sequence of sets of players $\{I_k\}_{1 \leq k \leq m}$ with the following properties:

(a) $I_1 = \{i\} \subset I_2 \subset \cdots \subset I_m \subset I_{k+1} \subset \cdots \subset I_{m+1} = S$.

(b) For every $k \in S$ there is a player $i_{k+1} \in \bigcup_{j \in I_k} \Gamma(j)$ with $I_k \setminus \{i_{k+1}\}$.

Therefore, on viewing the definition of $\sigma_i$ it suffices to show that for every $k = 1, 2, \ldots, m$, $\varphi^\infty_{j, i}(a) = \varphi^\infty_{j, i}(a_{k+1}) \prod_{j \in I_k} \varphi^\infty_{j, i}(a_j)$. We prove it by induction on $k$. For $k = 1$ the result is immediate. Suppose it is true for $k \geq 1$. We let $i_{k+1}$ be in $I_k$ such that $i_{k+1} \in \Gamma(j)$. Set $H_{i_{k+1}} := \{a_{i_{k+1}} = a_{i_{k+1}} \cap F_j \cap F_{i_{k+1}} \}$, It can be verified that $\mu([a = a] H_{i_{k+1}}) = \varphi^\infty_{j, i}(a_{k+1})$. Dividing $\mu(F_j \cap F_{i_{k+1}})$ yields that

$$\mu([a = a] F_j \cap F_{i_{k+1}}) = \varphi^\infty_{j, i}(a_{k+1}) \mu([a = a] H_{i_{k+1}}).$$

Thus $\varphi^\infty_{j, i}(a_{k+1}) = \varphi^\infty_{j, i}(a_{k+1}) \varphi^\infty_{j, i}(a_{i_{k+1}})$; then summing over $a_{i_{k+1}}$ we obtain $\varphi^\infty_{j, i}(a_{i_{k+1}}) = \varphi^\infty_{j, i}(a_{i_{k+1}}) \varphi^\infty_{j, i}(a_{i_{k+1}})$. It immediately follows from Proposition 1 that $\varphi^\infty_{j, i}(a_{i_{k+1}}) = \varphi^\infty_{j, i}(a_{i_{k+1}}) \varphi^\infty_{j, i}(a_{i_{k+1}})$, as required.

Furthermore we can observe that all the other players $i$ than $j$ agree on the same conjecture $\sigma_i(a_j) = \varphi^\infty_{j, i}(a_j)$ about $j$. We conclude that each action $a_i$ appearing with positive probability in $\sigma_i$ maximizes $g_i$ against the product of the distributions $\sigma_i$ with $i \neq j$. This implies that the profile $\sigma = (\sigma_i)_{i \in S}$ is a coalition $S$-Nash equilibrium of $G$, in completing the proof.

**5 Concluding remarks**

**Literature review**

Many authors have studied the learning processes modeled by Bayesian updating. The papers by E. Kalai and E. Lehrer [6] and J. S. Jordon [5] (and references therein) indicate increasing interest in the mutual learning processes in games that lead to equilibrium: Each player starts with initial erroneous belief regarding the actions of all the other players. If each player assigns a positive probability to the real action played by the others, their beliefs about the future actions of the others will converge in the long run.

E. Kalai and E. Lehrer [6] studies two-player repeated games, and they show the two strategies converge to an $\varepsilon$-mixed strategy Nash equilibrium of the repeated game if the common prior belief satisfies a certain uniform condition. J. S. Jordon [5] investigates the general convergence result for strategic form games. R. B. Myerson [13] proposes the Bayesian games with mediated communication in which each player is asked to confidentially report his type to the mediator, after getting these reports, the mediator confidentially recommends an action to each player. He characterizes the acceptable correlated equilibria as a subclass of the correlated equilibria in the Bayesian games.

As for as J.F. Nash's fundamental notion of strategic equilibrium is concerned, R.J. Aumann and A. Brandenburger [2] gives epistemic conditions for mixed strategy Nash equilibrium. They show that the
common-knowledge of the predictions of the players having the partitional information (that is, equivalent-ly, the S5-knowledge model) yields a Nash equilibri-
um of a game. The paper is in the line of Epis-temic Foundation of Interactive Decision Theory, starting by Aumann [1]. In the paper he introduced the formal notion of common-knowledge. However it is not clear just what learning process leads to the equilibrium.

To fill this gap from the epistemic point of view, Matsu-
hisa ([7], [9], [10]) presents his communication sys-
tem for a strategic game, which leads a mixed Nash equilibri-
um in several epistemic models. The articles [7], [9] [11] treats the communication system in the S4-knowledge model where each player communicates to other players by sending exact information about his/her conjecture on the others’ actions. In Matsuhisa and Strokan [11], the communication model in the p-belief system is introduced. Each player sends exact information that he/she believes that the others play their actions with probability at least his/her conjecture as messages. Matsuhisa [10] extended the com-
munication model to the case that the sending mes-
go are non-exact information that he/she believes that the others play their actions with probability at least his/her conjecture.

Conclusion

This paper points out that for coalition Nash equilib-
rium, common-knowledge cannot play such role as for mixed strategy Nash equilibrium. In fact, the profile of conjectures of a coalition may not yield a coalition Nash equilibrium even when the profile is commonly known among the all members of the coalition. To improve the situation we adopt the knowledge revisions model by Parikh and Krasucki [14] as a communication model. The main theorem in this paper shows that commu-
nication instead common-knowledge plays an essential role to form a coalition Nash equilibrium. We have ob-
served that in a communication process with revisions of players’ beliefs about the other actions among all the members in a coalition, their predictions induces a coalition Nash equilibrium of the game in the long run.

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