Optimization with Max-Min Fuzzy Relational Equations

Shu-Cherng Fang

Industrial Engineering and Operations Research
North Carolina State University
Raleigh, NC 27695-7906, U.S.A
www.ie.ncsu.edu/fangroup

October 31, 2008
ISORA’08 at Lijiang, China
Co-author: Pingke Li
Problem Facing

- Problem(P)

Minimize $f(x)$

s.t. $A \circ x = b$

$x \in [0,1]^n$

where $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is a function,

$A = (a_{ij})_{m \times n} \in [0,1]^{mn}$,

$b = (b_i)_{m \times 1} \in [0,1]^{m}$,

“$\circ$” is a matrix operation replacing “product” by “minimum” and “addition” by “maximum”, i.e.,

$$\max_{1 \leq j \leq n} \min(a_{ij}, x_j) = b_i, \text{ for } i = 1, \ldots, m.$$
Examples

1. Capacity Planning

\[ a_{ij} : \text{bandwidth in field from server} \ j \ \text{to user} \ i \]
\[ b_i : \text{bandwidth required by user} \ i \]
\[ x_j : \text{capacity of server} \ j \]

Consider

\[ \max_{1 \leq j \leq n} \ \min \ (a_{ij}, x_j) = b_i , \quad \text{for} \ i = 1, \ldots, m. \]
Examples

2. Fuzzy control / diagnosis / knowledge system

\[ a_{ij} \] : degree of input \( j \) relating to output \( i \)

\[ b_i \] : degree of output at state \( i \) (symptom)

\[ x_j \] : degree of input at state \( j \) (cause)

A fuzzy system is usually characterized by

\[
\max_{1 \leq j \leq n} t(a_{ij}, x_j) = b_i, \ \forall i, \ \text{or}
\]

\[
\min_{1 \leq j \leq n} s(a_{ij}, x_j) = b_i, \ \forall i,
\]

where "t" and "s" are triangular norms.
Triangular Norms


**t-norm:**

\[ t : [0,1] \times [0,1] \rightarrow [0,1] \] such that

1) \( t(x, y) = t(y, x) \) (commutative)

2) \( t(x, t(y, z)) = t(t(x, y), z) \) (associative)

3) \( t(x, y) \leq t(x, z) \), if \( y \leq z \) (monotonically nondecreasing)

4) \( t(x, 0) = 0 \) and \( t(x, 1) = x \) (boundary condition).

**s-norm (t co-norm):**

\[ s : [0,1] \times [0,1] \rightarrow [0,1] \] such that

\[ s(x, y) = 1 - t(1 - x, 1 - y) \quad \forall \ x, y \in [0,1] \]
### Triangular Norms

\[ t_w(\mu_A(x), \mu_B(x)) = \begin{cases} \min\{\mu_A(x), \mu_B(x)\} & \text{if } \max\{\mu_A(x), \mu_B(x)\} = 1 \\ 0, & \text{otherwise} \end{cases} \] (drastic product)

\[ s_w(\mu_A(x), \mu_B(x)) = \begin{cases} \max\{\mu_A(x), \mu_B(x)\} & \text{if } \min\{\mu_A(x), \mu_B(x)\} = 0 \\ 1, & \text{otherwise} \end{cases} \] (drastic sum)

\[ t_1(\mu_A(x), \mu_B(x)) = \max\{0, \mu_A(x) + \mu_B(x) - 1\} \] bounded difference

\[ s_1(\mu_A(x), \mu_B(x)) = \min\{1, \mu_A(x) + \mu_B(x)\} \] bounded sum

\[ t_{1.5}(\mu_A(x), \mu_B(x)) = \frac{\mu_A(x) \cdot \mu_B(x)}{2 - [\mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x)]} \] Einstein product

\[ s_{1.5}(\mu_A(x), \mu_B(x)) = \frac{\mu_A(x) + \mu_B(x)}{1 + \mu_A(x) \cdot \mu_B(x)} \] Einstein sum
Triangular Norms

\[ t_2(\mu_A(x), \mu_B(x)) = \mu_A(x) \cdot \mu_B(x) \]  
algebraic product

\[ s_2(\mu_A(x), \mu_B(x)) = \mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x) \]  
algebraic sum

\[ t_{2.5}(\mu_A(x), \mu_B(x)) = \frac{\mu_A(x) \cdot \mu_B(x)}{\mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x)} \]  
Hamacher product

\[ s_{2.5}(\mu_A(x), \mu_B(x)) = \frac{\mu_A(x) + \mu_B(x) - 2\mu_A(x) \cdot \mu_B(x)}{1 - \mu_A(x) \cdot \mu_B(x)} \]  
Hamacher sum

\[ t_3(\mu_A(x), \mu_B(x)) = \min \{\mu_A(x), \mu_B(x)\} \]  
minimum

\[ s_3(\mu_A(x), \mu_B(x)) = \max \{\mu_A(x), \mu_B(x)\} \]  
maximum

\[ t_w \leq \cdots \leq t_1 \cdots \leq t_2 \cdots \leq t_3 = \min \quad \leq s_3 = \max \cdots \leq s_2 \cdots \leq s_1 \cdots \leq s_w \]
Fuzzy Relational Equations

Given

\[ A = (a_{ij}) \in [0,1]^{m \times n}, \]
\[ b = (b_1, \ldots, b_m) \in [0,1]^m, \]

find

\[ x = (x_1, \ldots, x_n) \in [0,1]^n \quad \text{such that} \]

(max-t-norm composition \( A \circ x = b \))

\[ \max_{1 \leq j \leq n} t(a_{ij}, x_j) = b_i, \ \forall i. \]

(min-s-norm composition \( A \circ x = b \))

\[ \min_{1 \leq j \leq n} s(a_{ij}, x_j) = b_i, \ \forall i. \]

The solution set is denoted by \( \sum (A, b) \).
Difficulties in Solving Problem (P)

1. Algebraically, neither “maximum” nor “minimum” operations has an inverse operation.

\[
0.2x + 0.3 = 0.5 \quad \Rightarrow \quad x = \frac{0.5 - 0.3}{0.2} = 1
\]

\[
\max(0.3, \min(0.2, x)) = 0.5 \quad \Rightarrow \quad x = ?
\]

2. Geometrically, the solution set \( \sum (A, b) \) is a “combinatorially” generated “non-convex” set.
Solution Set of Max-t Equations

1. **Definition**: \( \hat{x} \in \Sigma(A, b) \) is a maximum solution if \( x \leq \hat{x}, \ \forall x \in \Sigma(A, b). \)

2. **Definition**: \( \check{x} \in \Sigma(A, b) \) is a minimum solution if \( x \geq \check{x}, \ \forall x \in \Sigma(A, b). \)

3. **Definition**: \( \hat{x} \in \Sigma(A, b) \) is a maximal solution if \( x \geq \hat{x} \) implies \( x = \hat{x}, \ \forall x \in \Sigma(A, b). \)

4. **Definition**: \( \check{x} \in \Sigma(A, b) \) is a minimal solution if \( x \leq \check{x} \) implies \( x = \check{x}, \ \forall x \in \Sigma(A, b). \)
Solution Set of Max-t Equations

• **Theorem:** For a continuous t-norm, if $\Sigma (A, b)$ is nonempty, then $\Sigma (A, b)$ can be completely determined by one maximum and a finite number of minimal solutions.

[Czogala / Drewhiak / Pedrycz (1982), Higashi / Klir (1984), di Nola (1985)]
Characteristics of Solution Sets

- Existence

[di Nola / Sessa / Pedrycz / Sanchez (1989)]

**Theorem**: For a continuous t-norm, \( \Sigma(A, b) \neq \phi \) if and only if it has a maximum solution \( \hat{x} = (\hat{x}_1, \ldots, \hat{x}_j) \)

with \( \hat{x}_j = \min_{1 \leq i \leq m} (a_{ij} \varphi b_i) \) where

\[
a \varphi b \equiv \sup \left\{ u \in [0,1] \mid t(a, u) \leq b \right\}.
\]
Characteristics of Solution Sets

• Uniqueness

• Complexity

  upper bound = $n^m$
Characteristics of Solution Sets

- **Theorem**: For a continuous s-norm, if $\Sigma (A, b)$ is nonempty, then $\Sigma (A, b)$ is completely determined by one minimum and a finite number of maximal solutions.

[Crown System]
Problem Facing

• Problem(P)

Minimize $f(x)$

s.t. $A \circ x = b$

$x \in [0,1]^m$

A nonconvex optimization problem over a region defined by a combinatorial number of vertices.
Optimization with Fuzzy Relation Equations

\[ f(x) = c^T x \quad \text{linear function} \]


**Lemma 1**: If \( c_j \leq 0 \) for all \( j \), then \( \hat{x} \) is an optimal solution.

**Lemma 2**: If \( c_j \geq 0 \) for all \( j \), then one of the minimal solutions is an optimal solution.
Optimization with Fuzzy Relation Equations

Theorem: Let

\[ c_j' = \begin{cases} c_j & \text{if } c_j \geq 0 \\ 0 & \text{if } c_j < 0 \end{cases} \]

and \( x^* = \begin{cases} \hat{x}_j & \text{if } c_j \geq 0 \\ \hat{x}_j & \text{if } c_j < 0 \end{cases} \),

where \( \hat{x}^* \) solves the problem with \( f(x) = (c')^T x \), then \( x^* \) is an optimal solution.

0-1 integer programming with a branch-and-bound solution technique.
Optimization with Fuzzy Relational Equations

• Extensions
  1. Objective function $f(x)$
     - linear fractional
     - geometric
     - general nonlinear
     - vector-valued
     - “max-t” or “min-s” operated

  2. Constraints
     - interval-valued
     - “max-t” or “min-s” operated

  3. Latticized optimization on the unit interval.
Theorem 1: Let $A \circ x = b$ be a consistent system of max-min (max-t) equations with a maximum solution $\hat{x}$.

The Problem (P) can be reduced to

\[
\begin{align*}
\text{Minimize} & \quad f(x) \\
\text{s. t.} & \quad Qu \geq e^m \\
(MIP) & \quad Gu \leq e^n \\
& \quad Vu \leq x \leq \hat{x} \\
& \quad u \in \{0,1\}^r
\end{align*}
\]

where $Q$ is $m$-by-$r$, $G$ is $n$-by-$r$, $V$ is $n$-by-$r$, $e^m$, $e^n$ are vectors of all ones, and $r$ is an integer (up to $m \times n$).
Major Results

**Theorem 2:** As in Theorem 1, if \( f(x) \) is linear, fractional linear, or monotone in each variable, then the Problem (P) can be further reduced to a 0-1 integer programming problem.

In particular, when the t-norm is Archimedean and \( f(x) \) is separable and monotone in each variable, then Problem (P) is equivalent to a “set covering problem”:

\[
\begin{align*}
\text{Minimize} & \quad \sum_j \left[ f_j(\hat{x}_j) - f_j(0) \right] u_j \\
\text{SCP} & \quad \text{s. t.} \quad Qu \geq e^m \\
& \quad u \in \{0,1\}^n.
\end{align*}
\]
Major Results

When the t-norm is non-Archimedean, then Problem (P) is equivalent to a “constrained set covering problem”.

Corollary: Problem (P) is in general NP-hard.

Theorem 3: In case \( f(x) = \max_{j \in N} f_j(x_j) \) with \( f_j(\cdot) \) being continuous and monotone for each \( j \), then Problem (P) can be solved in polynomial time.
Challenges Remain

• Efficient solution procedures to generate $\sum(A, b)$.

• Efficient algorithms for optimization problems with relational equation constraints.

• Efficient algorithms to generate an approximate solution of $\sum(A, b)$, i.e.,

$$\begin{align*}
\text{Minimize} & \quad \text{dist}(A \circ x, b) \\
\text{s. t.} & \quad x \in [0,1]^n.
\end{align*}$$
References


2. Li, P., Fang, S.-C., Minimizing a linear fractional function subject to a system of sup-$T$ equation with a continuous Archimedean triangular norm, to appear in *Journal of Systems Science and Complexity*.


4. Li, P., Fang, S.-C., Latticized linear optimization on the unit interval, submitted to *IEEE Transactions on Fuzzy System*. 
Thank you!

Questions?

<www.isc.nccsu.edu/fangroup>
<fang@eos.nccsu.edu>