Optimization with Max-Min Fuzzy Relational Equations

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Problem Facing

- Problem(P)

$$\text{Minimize} \quad f(x)$$
$$\text{s.t.} \quad A \circ x = b$$
$$x \in [0,1]^n$$

where $f: R^n \to R$ is a function,

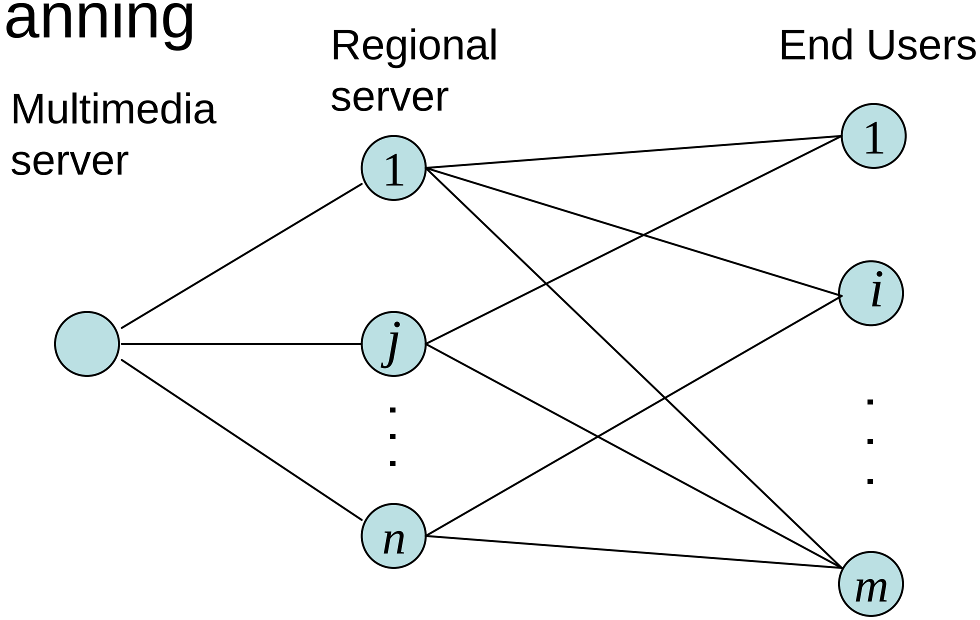
$$A = (a_{ij})_{m \times n} \in [0,1]^{mn},$$
$$b = (b_i)_{m \times 1} \in [0,1]^m,$$

"$\circ$" is a matrix operation replacing "product" by "minimum" and "addition" by "maximum", i.e.,

$$\max_{1 \le j \le n} \min(a_{ij}, x_j) = b_i, \text{ for } i = 1, \ldots, m.$$

Examples

1. Capacity Planning



a_{ij} : bandwidth in field from server j to user i

b_i : bandwidth required by user i

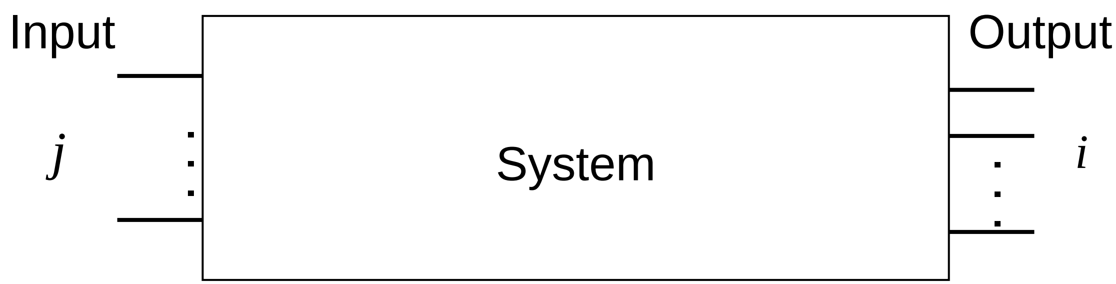
x_j : capacity of server j

Consider

$$\max_{1 \le j \le n} \min \ (a_{ij}, x_j) = b_i \ , \quad \text{for } i = 1, \ldots, m.$$

Examples

2. Fuzzy control / diagnosis / knowledge system



a_{ij} : degree of input j relating to output i

b_i : degree of output at state i (symptom)

x_j : degree of input at state j (cause)

A fuzzy system is usually characterized by

$$\max_{1 \le j \le n} t(a_{ij}, x_j) = b_i, \quad \forall i, \text{ or}$$

$$\min_{1 \le j \le n} s(a_{ij}, x_j) = b_i, \quad \forall i,$$

where "t" and "s" are triangular norms.

Triangular Norms

[Schweizer B. and Sklar A. (1961), “Associative functions and statistical triangle inequalities”, Mathematical Debrecen 8, 169-186.]

t-norm:

$$t:[0,1]\times[0,1]\to[0,1] \quad \text{such that}$$

1) $t(x,y)=t(y,x)$ (commutative)

2) $t(x,t(y,z))=t(t(x,y),z)$ (associative)

3) $t(x,y)\le t(x,z)$, if $y\le z$ (monotonically nondecreasing)

4) $t(x,0)=0$ and $t(x,1)=x$ (boundary condition).

s-norm (t co-norm):

$$s:[0,1]\times[0,1]\to[0,1] \quad \text{such that}$$

$$s(x,y)=1-t(1-x,1-y) \quad \forall\ x,y\in[0,1]$$

Triangular Norms

$$t_w(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = \begin{cases} \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\} & \text{if } \max\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\} = 1 \\ 0, \quad \text{otherwise} & \text{(drastic product)} \end{cases}$$

$$s_w(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = \begin{cases} \max\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\} & \text{if } \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\} = 0 \\ 1, \quad \text{otherwise} & \text{(drastic sum)} \end{cases}$$

$$t_1(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = \max\{0, \mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x) - 1\} \qquad \text{bounded difference}$$

$$s_1(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = \min\{1, \mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x)\} \qquad \text{bounded sum}$$

$$t_{1.5}(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = \frac{\mu_{\tilde{A}}(x) \cdot \mu_{\tilde{B}}(x)}{2 - [\mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x) - \mu_{\tilde{A}}(x) \cdot \mu_{\tilde{B}}(x)]} \qquad \text{Einstein product}$$

$$s_{1.5}(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = \frac{\mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x)}{1 + \mu_{\tilde{A}}(x) \cdot \mu_{\tilde{B}}(x)} \qquad \text{Einstein sum}$$

Triangular Norms

$$t_2(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = \mu_{\tilde{A}}(x) \cdot \mu_{\tilde{B}}(x)$$ algebraic product

$$s_2(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = \mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x) - \mu_{\tilde{A}}(x) \cdot \mu_{\tilde{B}}(x)$$ algebraic sum

$$t_{2.5}(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = \frac{\mu_{\tilde{A}}(x) \cdot \mu_{\tilde{B}}(x)}{\mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x) - \mu_{\tilde{A}}(x) \cdot \mu_{\tilde{B}}(x)}$$ Hamacher product

$$s_{2.5}(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = \frac{\mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x) - 2\mu_{\tilde{A}}(x) \cdot \mu_{\tilde{B}}(x)}{1 - \mu_{\tilde{A}}(x) \cdot \mu_{\tilde{B}}(x)}$$ Hamacher sum

$$t_3(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}$$ minimum

$$s_3(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = \max\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}$$ maximum

$$t_w \le \cdots \le t_1 \cdots \le t_2 \cdots \le t_3 = \min \quad \le s_3 = \max \cdots \le s_2 \cdots \le s_1 \cdots \le s_w$$

Fuzzy Relational Equations

Given

$$\mathrm{A} = (a_{ij}) \in [0,1]^{m \times n},$$

$$\mathrm{b} = (b_1, \ldots, b_m) \in [0,1]^m,$$

find

$$x = (x_1, \ldots, x_n) \in [0,1]^n \quad \text{such that}$$

(max-t-norm composition $A \circ x = b$)

$$\max_{1 \le j \le n} t(\mathrm{a}_{ij}, x_j) = b_i, \ \forall i.$$

(min-s-norm composition $A \circ x = b$)

$$\min_{1 \le j \le n} s(\mathrm{a}_{ij}, x_j) = b_i, \ \forall i.$$

The solution set is denoted by $\sum(A, b)$.

Difficulties in Solving Problem (P)

1. Algebraically, neither "maximum" nor "minimum" operations has an inverse operation.

$$0.2x + 0.3 = 0.5 \quad \Rightarrow \quad x = \frac{0.5 - 0.3}{0.2} = 1$$

$$\max\left(0.3, \min\left(0.2, x\right)\right) = 0.5 \ \Rightarrow \ x = ?$$

2. Geometrically, the solution set $\sum(A, b)$ is a "combinatorially" generated "non-convex" set.

Solution Set of Max-t Equations

1. Definition : $\hat{x} \in \Sigma(A,b)$ is a maximum solution

   if $x \le \hat{x}, \ \forall x \in \Sigma(A,b).$

2. Definition : $\check{x} \in \Sigma(A,b)$ is a minimum solution

   if $x \ge \check{x}, \ \forall x \in \Sigma(A,b).$

3. Definition : $\hat{x} \in \Sigma(A,b)$ is a maximal solution

   if $x \ge \hat{x}$ implies $x = \hat{x}, \ \forall x \in \Sigma(A,b).$

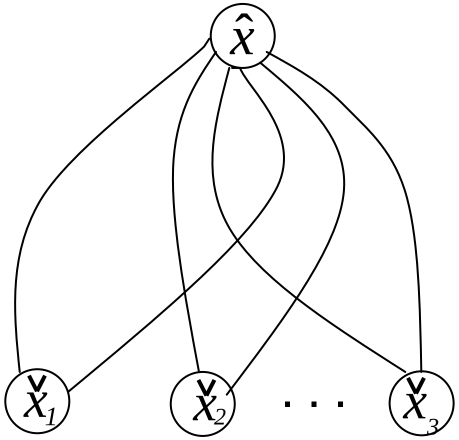
4. Definition : $\check{x} \in \Sigma(A,b)$ is a minimal solution

   if $x \le \check{x}$ implies $x = \check{x}, \ \forall x \in \Sigma(A,b).$

Solution Set of Max-t Equations

- Theorem: For a continuous t-norm, if $\Sigma(A, b)$ is nonempty, then $\Sigma(A, b)$ can be completely determined by one maximum and a finite number of minimal solutions.

[Czogala / Drewhiak / Pedrycz (1982), Higashi / Klir (1984), di Nola (1985)]



[Root System]

Characteristics of Solution Sets

- Existence

  [di Nola / Sessa / Pedrycz / Sanchez (1989)]

  Theorem: For a continuous t-norm, $\Sigma(A,b) \neq \phi$ if and only if it has a maximum solution $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_j)$ with $\hat{x}_j = \min_{1 \le i \le m} (a_{ij} \ \varphi \ b_i)$ where

$$a \ \varphi \ b \equiv \sup \left\{ u \in [0,1] \;\middle|\; t(a,u) \le b \right\}.$$

Characteristics of Solution Sets

- Uniqueness

[Sessa S. (1989), “Finite fuzzy relation equations with a unique solution in complete Brouwerian lattices,” Fuzzy Sets and Systems 29, 103-113.]

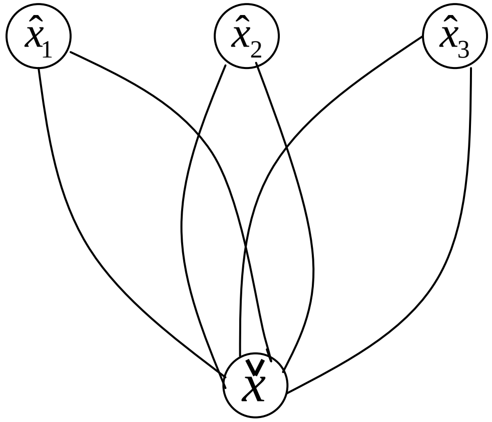
- Complexity

[Wang / Sessa/ di Nola/ Pedrycz (1984), “How many lower solutions does a fuzzy relation have?,” BUSEFAL 18, 67-74.]

upper bound = n^m

Characteristics of Solution Sets

- Theorem: For a continuous s-norm, if $\sum(A, b)$ is nonempty, then $\sum(A, b)$ is completely determined by one minimum and a finite number of maximal solutions.



[Crown System]

Problem Facing

- Problem(P)

$$\begin{aligned} \text{Minimize} \quad & f(x) \\ \text{s.t.} \quad & A \circ x = b \\ & x \in [0,1]^m \end{aligned}$$

A nonconvex optimization problem over a region defined by a combinatorial number of vertices.

Optimization with Fuzzy Relation Equations

- $f(x) = c^T x$ linear function

[Fang / Li (1999), "Solving fuzzy relation equations with a linear objective function, Fuzzy Sets and Systems 103, 107-113.]

Lemma1: If $c_j \leq 0$ for all j, then $\hat{x}$ is an optimal solution.

Lemma2: If $c_j \geq 0$ for all j, then one of the minimal solutions is an optimal solution.

Optimization with Fuzzy Relation Equations

Theorem : Let

$$c_j' = \begin{cases} c_j & if \quad c_j \geq 0 \\ 0 & if \quad c_j < 0 \end{cases} \quad \text{and} \quad x^* = \begin{cases} \check{x}_j^* & if \quad c_j \geq 0 \\ \hat{x}_j & if \quad c_j < 0 \end{cases},$$

where $\check{x}^*$ solves the problem with $f(x) = (c')^T x$,
then x^* is an optimal solution.

0-1 integer programming with a branch-and-bound solution technique.

Optimization with Fuzzy Relational Equations

- Extensions

1. Objective function $f(x)$

   - linear fractional
   - geometric
   - general nonlinear
   - vector-valued
   - “max-t” or “min-s” operated

2. Constraints

   - interval-valued
   - “max-t” or “min-s” operated

3. Latticized optimization on the unit interval.

Major Results

Theorem 1: Let $A \circ x = b$ be a consistent system of max-min (max-t) equations with a maximum solution $\hat{x}$.

The Problem (P) can be reduced to

$$
\text{(MIP)} \qquad
\begin{aligned}
\text{Minimize} \quad & f(x) \\
\text{s. t.} \quad & Qu \ge e^m \\
& Gu \le e^n \\
& Vu \le x \le \hat{x} \\
& u \in \{0,1\}^r
\end{aligned}
$$

where Q is m-by-r, G is n-by-r, V is n-by-r, e^m, e^n are vectors of all ones, and r is an integer (up to $m \times n$).

Major Results

Theorem 2: As in Theorem 1, if $f(x)$ is linear, fractional linear, or monotone in each variable, then the Problem (P) can be further reduced to a 0-1 integer programming problem.

In particular, when the t-norm is Archimedean and $f(x)$ is separable and monotone in each variable, then Problem (P) is equivalent to a "set covering problem":

$$
\text{(SCP)} \qquad \begin{aligned} \text{Minimize} \quad & \sum_j \left[f_j(\hat{x}_j) - f_j(0) \right] u_j \\ \text{s. t.} \quad & Qu \geq e^m \\ & u \in \{0,1\}^n . \end{aligned}
$$

Major Results

When the t-norm is non-Archimedean, then Problem (P) is equivalent to a "constrained set covering problem".

Corollary: Problem (P) is in general NP-hard.

Theorem 3: In case $f(x) = \max_{j \in N} f_j(x_j)$ with $f_j(\cdot)$ being continuous and monotone for each j, then Problem (P) can be solved in polynomial time.

Challenges Remain

- Efficient solution procedures to generate $\sum(A, b)$.

- Efficient algorithms for optimization problems with relational equation constraints.

- Efficient algorithms to generate an approximate solution of $\sum(A, b)$, i.e.,

$$\begin{array}{ll} \text{Minimize} & dist\left(A \circ x,\ b\right) \\ \text{s. t.} & x \in [0,1]^n. \end{array}$$

References

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3. Li, P., Fang, S.-C., A survey on fuzzy relational equations, Part I: Classification and solvability, to appear in *Fuzzy Optimization and Decision Making*.
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Thank you!

Questions?

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