Bifurcation of an Epidemic Model with Sub-optimal Immunity and Saturated Recovery Rate

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***Abstract*—In this paper, we study the bifurcation of an epidemic model with sub-optimal immunity and saturated treatment/recovery rate. Different from classical models, sub-optimal models are more realistic to explain the microparasite infections disease such as Pertussis and Influenza A. By carrying out the bifurcation analysis of the model, we show that for certain values of the model parameters, Hopf bifurcation, Bogdonov-Takens bifurcation and its associated homoclinic bifurcation occur. By studying the bifurcation curves, we can predict the persistence or extinction of diseases.**

Keywords- sub-optimal immunity; saturated treatment/recovery rate; Hopf bifurcation; homoclinic bifurcation; Bogdonov-Takens bifurcation;

I. INTRODUCTION

In recent years, extensive research has been carried out worldwide to develop more realistic epidemic models. For compartmental ODE models, several new models for incidence rate and treatment/recovery rate have been introduced. Subsequent analytical studies show that some of these epidemic models possess rich dynamics.

In a seminal paper, Ruan and Wang [1] presented a SIR epidemic model with the nonlinear incidence rate in the form of $\beta SI^2/(1+aI^2)$. In the paper, they consider a reduced system and perform an elaborative analysis of equilibrium through a quadratic equation. Using transformation to normal form, they show that the model undergoes Hopf bifurcation, homoclinic bifurcation and Bogdonov-Takens bifurcation. Following the paper, a few other papers discuss about the same dynamical behavior in the SIR model but with different forms of incidence rates such as $\beta SI/(1+aI+bI^2)$ [2] and $\beta SI^2/(1+aI+bI^2)$ [3].

Similarly, different treatment/recovery/removal rate are considered in order to predict the trend of disease transmission more accurately. Unlike the earlier model, the recent models may have two endemic equilibria when $R_0 < 1$. Hence, the eradication of disease depends not only on R_0, but also on the initial sizes of all sub-populations. The work in [4] is a pioneer work for bifurcation analysis which shows the existence of Hopf bifurcation and Bogdonov-Takens bifurcation for the model with constant removal rate. After the work of [4], various studies of bifurcation for the models with other treatment/recovery rate have been carried out. Backward bifurcation is shown in a SIR model with piecewise function treatment in [5], meanwhile the work in [6] claims the existence of Hopf bifurcation in a SIR model with saturated treatment rate. Furthermore, in the SIR model with saturated incidence rate and saturated treatment rate [7], only backward bifurcation is shown to exist, while reference [8] suggests that a SIS model with saturated recovery rate possesses Bogdonov-Takens bifurcation. However, to date, no analysis has been done to study the existence of Bogdonov-Takens bifurcation in the SIR model with saturated recovery rate. Hence, we intend to further study the bifurcation of the SIR model, and we will use the more generalized form of the model, namely the sub-optimal immunity model which lies in between the SIS and SIR models.

In this paper, we undertake the bifurcation analysis for an epidemic model with sub-optimal immunity and saturated treatment/recovery rate. Apart from using the saturated treatment/recovery rate, an additional parameter σ is used to form the sub-optimal immunity model as in [9]. The new model lies in between the SIS and SIR models. The sub-optimal immunity model will be more appropriate for the study of microparasite infections which usually occurs during childhood. After a primary infection, one may get temporary immunity (immune protection will wane over time) or partial immunity (immunity that may not fully protective). Examples of this kind of diseases include Pertussis (temporary immunity) and Influenza (partial immunity) [10]. Different to that in [9], we show in this paper that Bogdonov-Takens bifurcation and its associated homoclinic bifurcation exist in this sub-optimal immunity model.

Throughout the paper, for sake of simplicity, we choose some specific values for the parameters as [2] did. The parameter values can be easily replaced by other values as long as the conditions are fulfilled. Our analysis was carried out for the case where the basic reproduction number R_0 is less than unity. Apart from the discussion of Hopf bifurcation, we show that the sub-optimal immunity model undergoes Bogdonov-Takens bifurcation and its associated homoclinic bifurcation.

II. QUALITATIVE ANALYSIS

We consider a model with sub-optimal immunity and saturated recovery rate

$$\frac{dS}{dt} = A - \beta SI + \sigma T(I) - \mu S$$

$$\frac{dI}{dt} = \beta SI - T(I) - \mu I$$

$$\frac{dR}{dt} = (1-\sigma)T(I) - \mu R$$

(1)

where all the parameters are positive, A is the recruitment rate of susceptible population, β is the disease transmission rate, μ is the natural death rate and $T(I)$ is the recovery rate.

In our analysis at equilibrium point, we assume that $S + I + R = \frac{A}{\mu}$, and we take $T(I) = vI + \frac{cI}{1+aI}$ in which $c/(1+aI)$ and v are respectively the recovery rate of the infected population with and with no treatment.

. Defining the basic reproduction number by $R_0 = \frac{\beta A}{\mu(\mu + T'(0))}$, with $T(I) = vI + \frac{cI}{1+aI}$, we obtain $R_0 = \frac{\beta A}{\mu(\mu+v+c)}$. Define $R_1 = \frac{\beta Aa}{\beta k(v+c) + \beta\mu + \mu a(\mu+v)}$.

Now, we consider the following reduced system

$$\frac{dI}{dt} = \beta(\frac{A}{\mu} - I - R)I - vI - \frac{cI}{1+aI} - \mu I$$

$$\frac{dR}{dt} = k(vI + \frac{cI}{1+aI}) - \mu R \quad , \qquad k = 1-\sigma$$

(2)

At equilibrium, $\frac{dI}{dt} = 0$, $\frac{dR}{dt} = 0$, and hence from (2), we obtain $R = \frac{kI(v(1+aI)+c)}{\mu(1+aI)}$. Then by substituting this into the first equation in (2), after some algebra work, we obtain

$$[\beta a(kv+\mu)]I^2 + [\beta(kv+kc+\mu-Aa) + \mu a(\mu+v)]I$$
$$+\mu(\mu+v+c) - \beta A = 0$$

(3)

$$\text{Let } \Delta = [\beta(kv+kc+\mu-Aa) + \mu a(\mu+v)]^2$$
$$-4[\beta a(kv+\mu)][\mu(\mu+v+c) - \beta A] \quad .$$

(4)

Lemma 1.1

a) *System (2) has a unique positive equilibrium $E^*(I^*, R^*)$ under any of the following three conditions.*

i) $R_0 = 1$ *and* $[\beta(kv+kc+\mu-Aa) + \mu a(\mu+v)] < 0$, *for which* $I^* = \frac{-[\beta(kv+kc+\mu-Aa)+\mu a(\mu+v)]}{\beta a(kv+\mu)}$, $R^* = \frac{kI^*(v(1+aI^*)+c)}{\mu(1+aI^*)}$

ii) $R_0 > 1$, *for which*

$$I^* = \frac{-[\beta(kv+kc+\mu-Aa)+\mu a(\mu+v)]+\sqrt{\Delta}}{2[\beta a(kv+\mu)]}, \; R^* = \frac{kI^*(v(1+aI^*)+c)}{\mu(1+aI^*)}$$

iii) $\Delta = 0$ *and* $[\beta(kv+kc+\mu-Aa) + \mu a(\mu+v)] < 0$, *for which* $I^* = \frac{-[\beta(kv+kc+\mu-Aa)+\mu a(\mu+v)]}{2[\beta a(kv+\mu)]}$, $R^* = \frac{kI^*(v(1+aI^*)+c)}{\mu(1+aI^*)}$

b) *System (2) has two positive equilibria $E_1(I_1, R_1)$ and $E_2(I_2, R_2)$ if and only if*

$R_0 < 1$, $\Delta > 0$ *and* $[\beta(kv+kc+\mu-Aa) + \mu a(\mu+v)] < 0$, *where*

$$I_1 = \frac{-[\beta(kv+kc+\mu-Aa)+\mu a(\mu+v)]-\sqrt{\Delta}}{2[\beta a(kv+\mu)]}, \; R_1 = \frac{kI_1(v(1+aI_1)+c)}{\mu(1+aI_1)}$$

$$I_2 = \frac{-[\beta(kv+kc+\mu-Aa)+\mu a(\mu+v)]+\sqrt{\Delta}}{2[\beta a(kv+\mu)]}, \; R_2 = \frac{kI_2(v(1+aI_2)+c)}{\mu(1+aI_2)}$$

The Jacobian matrix for system (2) is

$$M = \begin{bmatrix} -\beta I + \beta(\frac{A}{\mu} - I - R) - v - \frac{c}{1+aI} + \frac{caI}{(1+aI)^2} - \mu & -\beta I \\ k\left(v + \frac{c}{1+aI} - \frac{caI}{(1+aI)^2}\right) & -\mu \end{bmatrix}$$

The determinant of M is as follows

$$\det(M) = \frac{1}{(1+aI)^2}\Big(2\beta a^2(kv+\mu)I^3$$
$$+\big(\beta a(4kv+kc+4\mu-Aa) + \mu a^2(\mu+v)\big)I^2$$
$$+\big(2\beta(\mu-Aa+kc+kv) + 2\mu a(v+\mu)\big)I$$
$$+\mu(\mu+v+c) - \beta A\Big)$$

The sign of the determinant is determined by the sign of

$$S_1 = 2\beta a^2(kv+\mu)I^3$$
$$+\big(\beta a(4kv+kc+4\mu-Aa) + \mu a^2(\mu+v)\big)I^2$$
$$+\big(2\beta(\mu-Aa+kc+kv) + 2\mu a(v+\mu)\big)I$$
$$+\mu(\mu+v+c) - \beta A.$$

Using (3), we get

$$S_1 = \big(\beta a(Aa+2kv-kc+2\mu) - \mu a^2(\mu+v)\big)I^2$$
$$+\big(2\beta(\mu+kc+kv) - 2c\mu a\big)I + \mu(\mu+v+c) - \beta A$$

Lemma 1.2

a) *The unique positive equilibrium $E^*(I^*, R^*)$ in system (2) is*

i) *a degenerate equilibrium if* $\Delta = 0$, $[\beta(kv+kc+\mu-Aa) + \mu a(\mu+v)] < 0$.

ii) *a center-type equilibrium if* $R_0 > 1$ *while* $\operatorname{tr}(M) = 0$

b) *The positive equilibrium $E_1(I_1, R_1)$ in system (2) leads to $S_1(I_1) < 0$ while $\Delta > 0$, $R_0 < 1$ and $[\beta(kv+kc+\mu-Aa) + \mu a(\mu+v)] < 0$. It is thus a saddle point.*

c) *The positive equilibrium $E_2(I_2, R_2)$ in system (2) leads to $S_1(I_2) > 0$ while $\Delta > 0$, $R_0 < 1$ and*

$[\beta(kv+kc+\mu-Aa)+\mu a(\mu+v)]<0$. *It is thus a node, focus or center.*

III. HOPF BIFURCATION

In this section, we will show that the model in (2) undergoes Hopf bifurcation for some values.

Let $(\beta,v,c,a,\mu,k)=(\frac{1}{2},8,8,3,1,\frac{1}{4})$ for (I_2,R_2) and set $\mathrm{tr}(M)=0$, then we obtain $A=\frac{51}{2}$ while $(I_2,R_2)=(1,\frac{5}{2})$. This happens when $R_0=\frac{3}{4}<1$.

Replacing *I* and *R* by *x* and *y*, namely $(I_2,R_2)=(x_2,y_2)$, we have

$$\frac{dx}{dt}=\frac{1}{2}\left(\frac{51}{2}-x-y\right)x-9x-\frac{8x}{1+3x}$$

$$\frac{dy}{dt}=2x+\frac{2x}{1+3x}-y$$

(5)

To translate (x_2,y_2) to the origin, we set $X=x-1$, $Y=y-\frac{5}{2}$ and rename *X,Y* as *x,y* respectively. Then

$$\frac{dx}{dt}=\frac{1}{2}\left(\frac{51}{2}-(x+1)-(y+\frac{5}{2})\right)(x+1)-9(x+1)-\frac{8(x+1)}{1+3(x+1)}$$

$$\frac{dy}{dt}=2(x+1)+\frac{2(x+1)}{1+3(x+1)}-(y+\frac{5}{2})$$

(6)

Using the Taylor expansion for (6), we have

$$\frac{dx}{dt}=-\frac{1}{2}y+(1-\frac{1}{2}y)x-\frac{1}{8}x^2-\frac{9}{32}x^3+\frac{27}{128}x^4+O(|x,y|^5)$$

$$\frac{dy}{dt}=-y+\frac{17}{8}x-\frac{3}{32}x^2+\frac{9}{128}x^3-\frac{27}{512}x^4+O(|x,y|^5)$$

(7)

The Jacobian matrix for (7) at (x_2,y_2) is

$$M=\begin{bmatrix}1 & -\frac{1}{2}\\ \frac{17}{8} & -1\end{bmatrix}.$$

We thus have $\mathrm{tr}(M)=0$ and $\det(A)=\frac{1}{16}>0$, and Hopf bifurcation occurs.

By carrying out transformation $X=x$, $Y=x-\frac{1}{2}y$, and then renaming *X,Y* as *x,y* respectively, (7) becomes

$$\frac{dx}{dt}=y-\frac{9}{8}x^2+xy-\frac{9}{32}x^3+\frac{27}{128}x^4+O(|x,y|^5)$$

$$\frac{dy}{dt}=-\frac{1}{16}x-\frac{69}{64}x^2+xy-\frac{81}{256}x^3+\frac{243}{1024}x^4+O(|x,y|^5)$$

(8)

Making the change of variables $u=-x$, $v=4y$, we obtain

$$\frac{du}{dt}=-\frac{1}{4}v+\frac{9}{8}u^2+\frac{1}{4}uv-\frac{9}{32}u^3-\frac{27}{128}u^4+O(|u,v|^5)$$

$$\frac{dv}{dt}=\frac{1}{4}u-\frac{69}{64}u^2-uv+\frac{81}{64}u^3+\frac{243}{256}u^4+O(|u,v|^5)$$

(9)

Let $k_1=\frac{1}{16}$ and

$$F_1(u,v)=\frac{9}{8}u^2+\frac{1}{4}uv-\frac{9}{32}u^3-\frac{27}{128}u^4+O(|u,v|^5)$$

$$F_2(u,v)=-\frac{69}{64}u^2-uv+\frac{81}{64}u^3+\frac{243}{256}u^4+O(|u,v|^5).$$

(10)

We can get the first Liapunov constant, σ, by

$$\sigma=\frac{1}{16}\left[\frac{\partial^3F_1}{\partial u^3}+\frac{\partial^3F_1}{\partial u\partial v^2}+\frac{\partial^3F_2}{\partial u^2\partial v}+\frac{\partial^3F_2}{\partial v^3}\right]$$

$$+\frac{1}{16\sqrt{k_1}}\left[\frac{\partial^2F_1}{\partial u\partial v}\left(\frac{\partial^2F_1}{\partial u^2}+\frac{\partial^2F_1}{\partial v^2}\right)-\frac{\partial^2F_2}{\partial u\partial v}\left(\frac{\partial^2F_2}{\partial u^2}+\frac{\partial^2F_2}{\partial v^2}\right)-\frac{\partial^2F_1}{\partial u^2}\frac{\partial^2F_2}{\partial u^2}+\frac{\partial^2F_1}{\partial v^2}\frac{\partial^2F_2}{\partial v^2}\right]$$

$$=\frac{6465}{8192}.$$

Hence, there is an unstable periodic orbit when *A* increases from $\frac{51}{2}$.

In the following, we choose *A* as a bifurcation parameter. Let $A=\frac{51}{2}+\varepsilon$. From (5), we obtain,

$$\frac{dx}{dt}=\frac{1}{2}\left(\frac{51}{2}+\varepsilon-x-y\right)x-9x-\frac{8x}{1+3x}$$

$$\frac{dy}{dt}=2x+\frac{2x}{1+3x}-y$$

(11)

It is easy to show that

$$(x_2^*,y_2^*)=\left(\frac{35}{36}+\frac{\varepsilon}{6}+\frac{\sqrt{36\varepsilon^2+564\varepsilon+1}}{36},\frac{2021}{828}+\frac{43\varepsilon}{138}+\frac{49\sqrt{36\varepsilon^2+564\varepsilon+1}}{828}\right)$$

is the positive equilibrium of the system (11). The Jacobian matrix is given by

$$M=\begin{bmatrix}M_{11} & -\frac{x_2^*}{2}\\ M_{21} & -1\end{bmatrix}\text{ where}$$

$$M_{11}=-x_2^*+\frac{15}{4}+\frac{\varepsilon}{2}-\frac{y_2^*}{2}-\frac{8}{1+3x_2^*}+\frac{24x_2^*}{(1+3x_2^*)^2}$$

$$M_{21}=2+\frac{2}{1+3x_2^*}-\frac{6x_2^*}{(1+3x_2^*)^2}.$$

Hence, the characteristic equation is given by

$$\lambda^2+(1-M_{11})\lambda-M_{11}+\frac{x_2^*}{2}M_{21}=0.$$

We thus obtain $\lambda=\frac{m_A\pm\sqrt{m_B}}{m_C}$ where

$$m_A=1339-36378\varepsilon-6156\varepsilon^2-216\varepsilon^3$$
$$+(-1339-744\varepsilon-36\varepsilon^2)\sqrt{1+564\varepsilon+36\varepsilon^2}$$

$$m_B=-32696110-19572948792\varepsilon-13313554632\varepsilon^2$$

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$$-2408256576\varepsilon^3-149198112\varepsilon^4-1399680\varepsilon^5+93312\varepsilon^6$$
$$+(-397285586-826878324\varepsilon-246740976\varepsilon^2$$
$$-21607776\varepsilon^3-355104\varepsilon^4+15552\varepsilon^5)\sqrt{1+564\varepsilon+36\varepsilon^2}$$

$$m_C=72(1105+564\varepsilon+36\varepsilon^2+(47+6\varepsilon)\sqrt{1+564\varepsilon+36\varepsilon^2})$$

Hence, we have

i) $\mathrm{Re}\,\lambda(\varepsilon)=0$ when $\varepsilon=0$.

ii) $\mathrm{Im}\,\lambda(\varepsilon)=\frac{\sqrt{429981696}}{82944}\neq 0$ when $\varepsilon=0$.

iii) $\mathrm{Re}\frac{d}{d\varepsilon}\lambda(\varepsilon)=-5\neq 0$ when $\varepsilon=0$.

Theorem 3.1 *There exist a $\sigma_1>0$ and a function $\varepsilon=\varepsilon(x_1)$ defined on $0<x_1-1\le\sigma_1$, which satisfy $\varepsilon(1)=0$ and when $\varepsilon=\varepsilon(x_1)<0$, system (11) has a unique unstable limit cycle which passes through $(x_1,\frac{5}{2})$.*

Fig. 1 shows an unstable orbit for system (2) when $(\beta,v,c,a,\mu,k)=(\frac{1}{2},8,8,3,1,\frac{1}{4})$ and $A=25.52$.

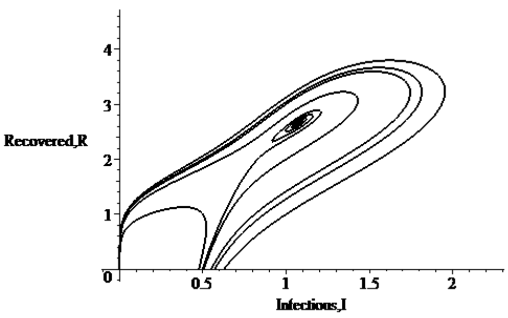

Figure 1. An unstable periodic orbit when $(\beta,v,c,a,\mu,k)=(\frac{1}{2},8,8,3,1,\frac{1}{4})$ and $A=25.52$.

IV. BOGDANOV-TAKENS BIFURCATION

In this section, we will study the Bogdanov-Takens bifurcation for some values of the model in (2).

We choose $(\beta,v,a,\mu,k)=(\frac{1}{2},2,\frac{1}{2},1,\frac{1}{4})$ for (I^*,R^*) and let $\Delta=0$, and we obtain $c=8$. Setting $A=19$, we obtain $(I^*,R^*)=(2,3)$, $\mathrm{trace}(M)=0$ and $\det(M)=0$.

Writing *I* and *R* as *x* and *y*, namely $(I^*,R^*)=(x^*,y^*)$, we have

$$\frac{dx}{dt}=\frac{1}{2}(19-x-y)x-3x-\frac{8x}{1+\frac{1}{2}x}$$

$$\frac{dy}{dt}=\frac{1}{2}x+\frac{2x}{1+\frac{1}{2}x}-y$$

(12)

To translate (x^*,y^*) to the origin, we set $X=x-2$, $Y=y-3$ and rename *X,Y* as *x,y* respectively. Then

$$\frac{dx}{dt}=\frac{1}{2}(19-(x+2)-(y+3))(x+2)-3(x+2)-\frac{8(x+2)}{1+\frac{1}{2}(x+2)}$$

$$\frac{dy}{dt}=\frac{1}{2}(x+2)+\frac{2(x+2)}{1+\frac{1}{2}(x+2)}-(y+3)$$

(13)

Using the Taylor expansion for (13), we have

$$\frac{dx}{dt}=-y+(1-\frac{1}{2}y)x-\frac{1}{8}x^3+\frac{1}{32}x^4+O(|x,y|^5)$$

$$\frac{dy}{dt}=-y+x-\frac{1}{8}x^2+\frac{1}{32}x^3-\frac{1}{128}x^4+O(|x,y|^5)$$

(14)

The Jacobian matrix for (14) at (x^*,y^*) is

$$M=\begin{bmatrix}1 & -1\\ 1 & -1\end{bmatrix}$$

We thus have $\mathrm{tr}(M)=0$ and $\det(A)=0$. Clearly, the matrix *M* has two zero eigenvalues, and thus the Bogdanov-Takens bifurcation occurs.

By carrying out transformation $X=x$, $Y=x-y$, and renaming *X,Y* as *x,y* respectively, (14) becomes

$$\frac{dx}{dt}=y-\frac{1}{2}x^2+\frac{1}{2}xy-\frac{1}{8}x^3+\frac{1}{32}x^4+O(|x,y|^5)$$

$$\frac{dy}{dt}=-\frac{3}{8}x^2+\frac{1}{2}xy-\frac{5}{32}x^3+\frac{5}{128}x^4+O(|x,y|^5)$$

(15)

In order to obtain the canonical normal form, we follow the procedure as in [11]. Setting $u=x-\frac{1}{4}x^2$, $v=y-\frac{1}{2}y^2$, we obtain

$$\frac{du}{dt}=v+O(|u,v|^3)$$

$$\frac{dv}{dt}=\frac{1}{2}uv-\frac{3}{8}u^2+O(|u,v|^3)$$

(16)

In the following, we find the universal unfolding of $(I^*,R^*)=(2,3)$ by choosing parameters *A* and *c* as bifurcation parameters in a small neighbourbood of $(\beta,v,a,\mu,k)=(\frac{1}{2},2,\frac{1}{2},1,\frac{1}{4})$. Let $A=19+\lambda_1$ and $c=8+\lambda_2$. We have

$$\frac{dx}{dt}=\frac{1}{2}(19+\lambda_1-x-y)x-3x-\frac{(8+\lambda_2)x}{1+\frac{1}{2}x},$$

$$\frac{dy}{dt}=\frac{1}{2}x+\frac{(8+\lambda_2)x}{4(1+\frac{1}{2}x)}-y$$

(17)

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To translate (x^*, y^*) to the origin, we set $X = x - 2$, $Y = y - 3$ and rename X,Y as x,y respectively. Then

$$\frac{dx}{dt} = \frac{1}{2}\big((19+\lambda_1) - (x+2) - (y+3)\big)(x+2) - 3(x+2) - \frac{(8+\lambda_2)(x+2)}{1+\frac{1}{2}(x+2)}$$

$$\frac{dy}{dt} = \frac{1}{2}(x+2) + \frac{(8+\lambda_2)(x+2)}{4\left(1+\frac{1}{2}(x+2)\right)} - (y+3)$$

(18)

Using the Taylor expansion for (18), we have

$$\frac{dx}{dt} = \lambda_1 - \lambda_2 - y + (-\tfrac{1}{4}\lambda_2 + \tfrac{1}{2}\lambda_1 + 1 - \tfrac{1}{2}y)x + \tfrac{1}{16}\lambda_2 x^2 + (-\tfrac{1}{8} - \tfrac{1}{64}\lambda_2)x^3 + (\tfrac{1}{32} + \tfrac{1}{256}\lambda_2)x^4 + O(|x,y|^5)$$

$$\frac{dy}{dt} = \tfrac{1}{4}\lambda_2 - y + (\tfrac{1}{16}\lambda_2 + 1)x + (-\tfrac{1}{8} - \tfrac{1}{16}\lambda_2)x^2 + (\tfrac{1}{32} + \tfrac{1}{256}\lambda_2)x^3 + (-\tfrac{1}{128} - \tfrac{1}{1024}\lambda_2)x^4 + O(|x,y|^5) \quad (19)$$

Let $X = x$,

$$Y = \lambda_1 - \lambda_2 - y + (-\tfrac{1}{4}\lambda_2 + \tfrac{1}{2}\lambda_1 + 1 - \tfrac{1}{2}y)x + \tfrac{1}{16}\lambda_2 x^2 + (-\tfrac{1}{8} - \tfrac{1}{64}\lambda_2)x^3 + (\tfrac{1}{32} + \tfrac{1}{256}\lambda_2)x^4 + O(|x,y|^5)$$

and rename X,Y as x,y respectively. Then we obtain

$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = a_0 + a_1 x + a_2 y + a_3 x^2 + a_4 xy + a_5 y^2 + O(|x,y,\lambda|^3)$$

where $a_0 = \lambda_1 - \frac{5}{4}\lambda_2$, $a_1 = \frac{1}{2}\lambda_1 - \frac{7}{16}\lambda_2$, $a_2 = \frac{1}{4}\lambda_2$, $a_3 = -\frac{3}{8}$, $a_4 = -\frac{1}{2}$ and $a_5 = \frac{1}{2}$.

By setting $X = x + \frac{a_2}{a_4}$ (i.e. $X = x - \frac{1}{2}\lambda_2$) and rewriting X as x, we have

$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = b_0 + b_1 x + a_3 x^2 + a_4 xy + a_5 y^2 + O(|x,y,\lambda|^3)$$

where $b_0 = \lambda_1 - \frac{5}{4}\lambda_2 + \frac{1}{4}\lambda_1\lambda_2 - \frac{5}{16}\lambda_2^2$, $b_1 = \frac{1}{2}\lambda_1 - \frac{13}{16}\lambda_2$, $a_3 = -\frac{3}{8}$, $a_4 = -\frac{1}{2}$ and $a_5 = \frac{1}{2}$.

By rewriting the equation using the new time τ with $dt = (1 - a_5 x)d\tau$ (i.e. $dt = (1 - \frac{1}{2}x)d\tau$) and then rewriting τ as t, we obtain

$$\frac{dx}{dt} = y(1 - \tfrac{1}{2}x)$$

$$\frac{dy}{dt} = (1 - \tfrac{1}{2}x)\left(b_0 + b_1 x + a_3 x^2 + a_4 xy + a_5 y^2 + O(|x,y,\lambda|^3)\right)$$

Carrying out the transformation $X = x$, $Y = y(1 - \frac{1}{2}x)$, and then renaming X,Y as x,y respectively, we have

$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = b_0 + c_1 x + c_2 x^2 + a_4 xy + O(|x,y,\lambda|^3)$$

where $b_0 = \lambda_1 - \frac{5}{4}\lambda_2 + \frac{1}{4}\lambda_1\lambda_2 - \frac{5}{16}\lambda_2^2$, $c_1 = -\frac{1}{2}\lambda_1 + \frac{7}{16}\lambda_2$, $c_2 = -\frac{3}{8}$ and $a_4 = -\frac{1}{2}$.

By the change of variables $X = \frac{a_4^2}{c_2}x$, $Y = \frac{a_4^3}{c_2^2}y$, $\tau = \frac{c_2}{a_4}t$, and then renaming X, Y, τ as x, y, t respectively, we obtain

$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = \tau_1 + \tau_2 x + x^2 + xy + O(|x,y,\lambda|^3)$$

where $\tau_1 = \frac{b_0 a_4^2}{c_2^3}$ and $\tau_2 = \frac{c_1 a_4^2}{c_2^2}$.

By putting $\tau_1 = \frac{1}{4}\tau_2^2$ and simplifying it, system (12) has a saddle-node bifurcation, and the saddle-node bifurcation curve is given by $-384\lambda_1 + 480\lambda_2 + 16\lambda_1\lambda_2 - 64\lambda_1^2 + 71\lambda_2^2 = 0$

Theorem 4.1 *At the Bogdanov point, the model (2) with* $(\beta, v, a, \mu, k) = (\frac{1}{2}, 2, \frac{1}{2}, 1, \frac{1}{4})$, $A = 19$ *and* $c = 8$, *in a small neighbourhood of* $(I^*, R^*) = (2,3)$, *has the following bifurcation :*

i) saddle-node bifurcation: the saddle-node bifurcation curve is given by

$$-384\lambda_1 + 480\lambda_2 + 16\lambda_1\lambda_2 - 64\lambda_1^2 + 71\lambda_2^2 + O(|\lambda|^2) = 0,$$

ii) Hopf bifurcation : the Hopf bifurcation curve is given by

$$16\lambda_1 - 20\lambda_2 + 4\lambda_1\lambda_2 - 5\lambda_2^2 + O(|\lambda|^2) = 0,$$

iii) Homoclinic bifurcation : the homoclinic bifurcation curve is given by

$$-600\lambda_1 + 750\lambda_2 - 318\lambda_1\lambda_2 + 96\lambda_1^2 + 261\lambda_2^2 + O(|\lambda|^2) = 0.$$

Fig. 2 shows the homoclinic bifurcation when $\lambda_1 = 0.05$, $\lambda_2 = 0.03997138969$ for system (17).

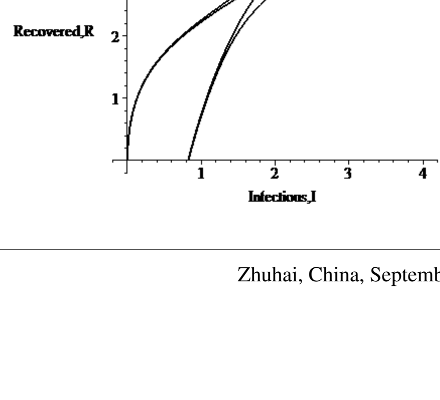

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Figure 2. Homoclinic bifurcation when $\lambda_1 = 0.05$, $\lambda_2 = 0.03997138969$

From the result in Theorem 4.1, we study the bifurcation curves near the origin on the (λ_1, λ_2) plane. The curves pass through the origin and there are four regions separated by these bifurcation curves. If we take near $\lambda_1 = 0.2$, we obtain the region as in Figure 3.

The Jacobian matrix for system (17) is

$$M = \begin{bmatrix} M_{11} & \frac{-x}{2} \\ M_{21} & -1 \end{bmatrix} \text{ where}$$

$$M_{11} = \frac{1}{2(2+x)^2}\left(-2x^3 + (5+\lambda_1 - y)x^2 + (4\lambda_1 + 44 - 4y)x - 8\lambda_2 - 12 - 4y\right), \text{ and}$$

$$M_{12} = \frac{1}{2(2+x)^2}\left(x^2 + 4x + 2\lambda_2 + 20\right).$$

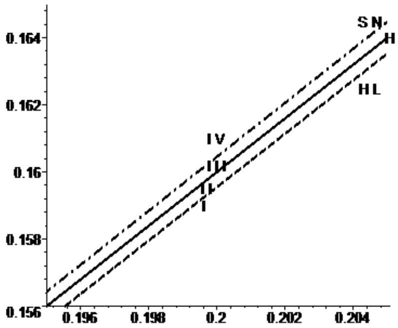

Figure 3. The four typical regions separated by the bifurcation curves. The horizontal axis is the λ_1 -axis and the vertical axis is the λ_2 -axis

If we take $\lambda_1 = 0.2$, after some simple calculation, we obtain the result as shown in the Table 1 below.

TABLE I. THE CLASSIFICATION OF EQUILIBRIUM POINTS

	λ_2	E_i	det(M)	tr(M)	Q	Conclusion
I	0.1590	E_1	(-)	(+)	(+)	Unstable saddle
		E_2	(+)	(-)	(-)	Stable focus
II	0.1596	E_1	(-)	(+)	(+)	Unstable saddle
		E_2	(+)	(-)	(-)	Stable focus
III	0.1602	E_1	(-)	(+)	(+)	Unstable saddle
		E_2	(+)	(+)	(-)	Unstable focus
IV	0.1610	No positive equilibrium				

$Q = (\text{tr}(M))^2 - 4(\det(M))$

When (λ_1, λ_2) lies in region I as in Figure 3, there is no limit cycle or homoclinic orbit and E_2 is a stable focus. If (λ_1, λ_2) lies in region II, there is a unique limit cycle inside the positive orbits of system (17) and the orbits approach E_2 as t tends to infinity. In this situation, the disease is persistent inside the cycle. When (λ_1, λ_2) lies in region III, E_2 becomes an unstable focus and the limit cycle disappears. In this stage, at finite time, any positive orbits, except for the two equilibria E_1 and E_2, will tend to the axis $R = 0$, i.e. the disease becomes extinct. When (λ_1, λ_2) lies in region IV, there is no positive equilibrium and the disease will disappear. The classification of the equilibrium points can be easily checked by the eigenvalues of the Jacobian matrix, *M*.

V. CONCLUSION

In this paper, we have proposed an epidemic model with sub-optimal immunity and saturated treatment/recovery rate. Through global analysis, the system in (2) has been shown to have rich dynamical behaviour including Hopf bifurcation, Bogdonov-Takens bifurcation and its associated homoclinic bifurcation. We also show that when the bifurcation parameters are within certain regions, the disease will be persistent or extinct.

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